University of Amsterdam

AMLAB Amsterdam Machine Learning Lab

Group Equivariant Deep Learning

Lecture 2 - Steerable group convolutions

Lecture 2.3 - Group Theory | Irreducible representations and Fourier trafo

Preliminaries for steerable feature fields and steerable g-conv intuition

With a focus on SO(2)

Erik Bekkers, Amsterdam Machine Learning Lab, University of Amsterdam

This mini-course serves as a module with the UvA Master AI course Deep Learning 2 https://uvadl2c.github.io/

Equivalence of group representations

Two representations $\rho^A(g)$ and $\rho^B(g)$ are said to be **equivalent** if they relate via a similarity transform

$$\rho^B(g) = Q^{-1}\,\rho^A(g)\,Q$$

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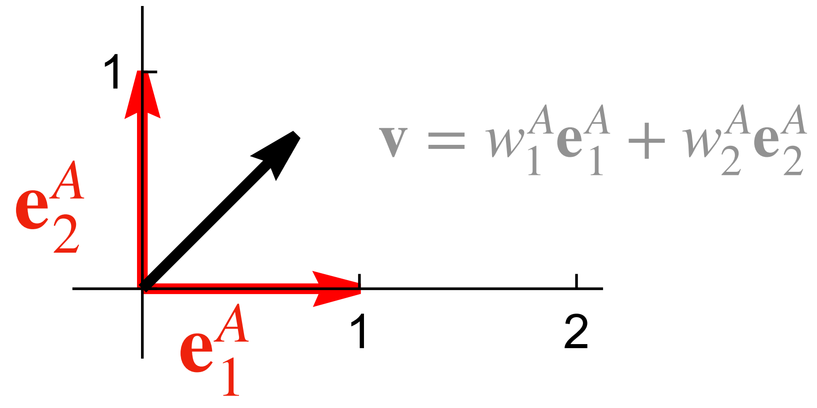
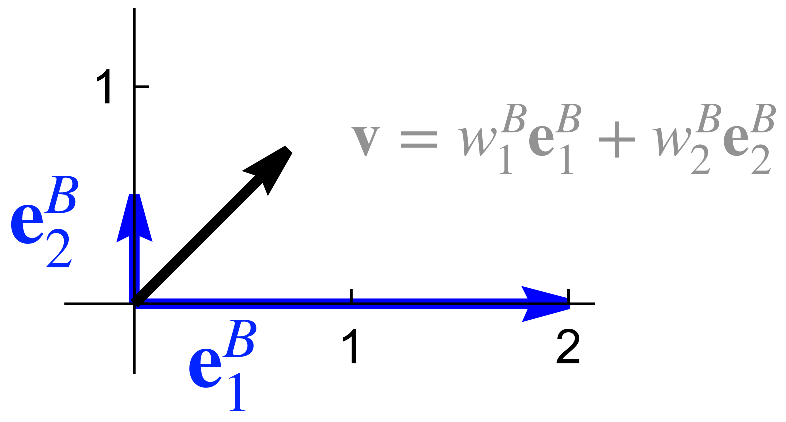
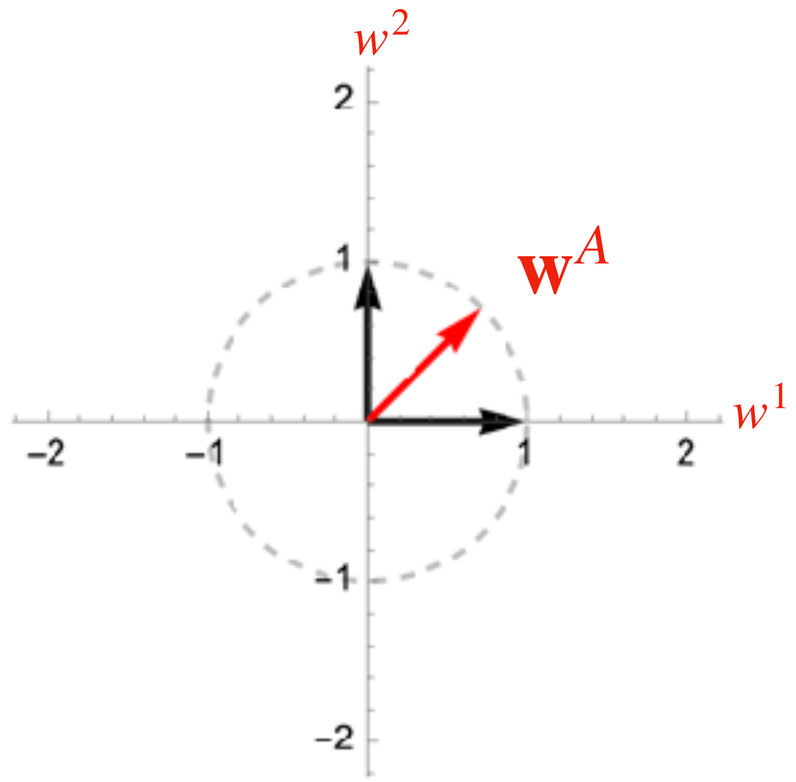
Standard basis for $\mathbb{R}^2$



Scaled basis $Q = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = (\mathbf{e}_1^B, \mathbf{e}_2^B)$





$$\mathbf{w}^A = Q\,\mathbf{w}^B$$

$$\longleftarrow$$

$$\longrightarrow$$

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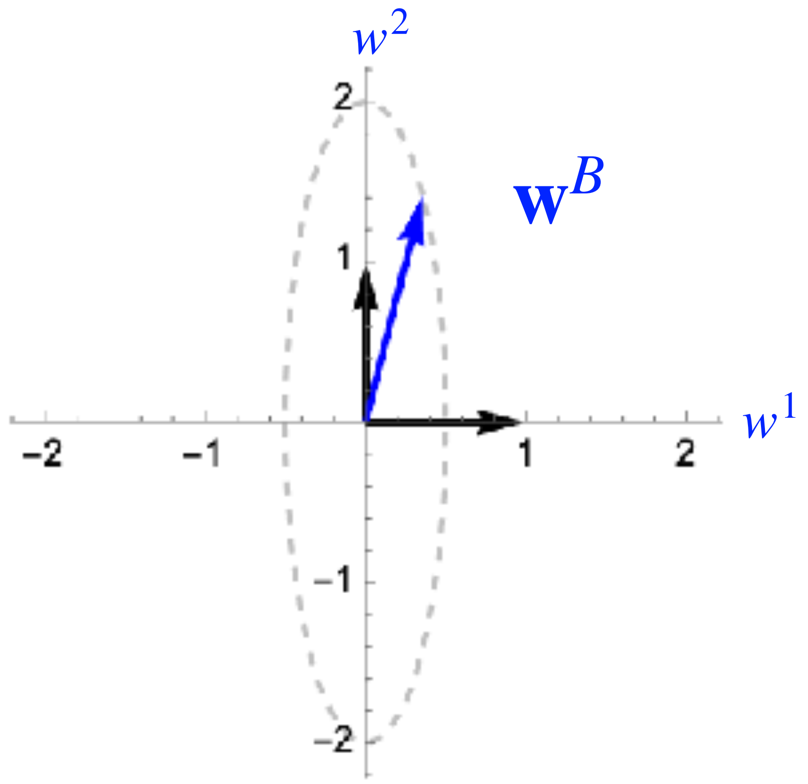

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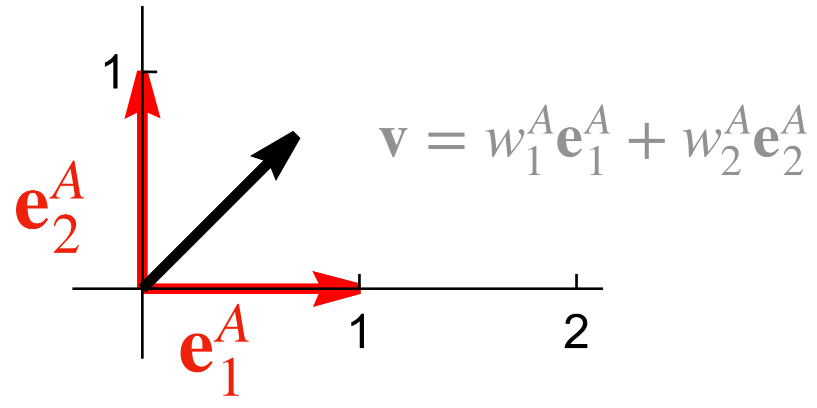

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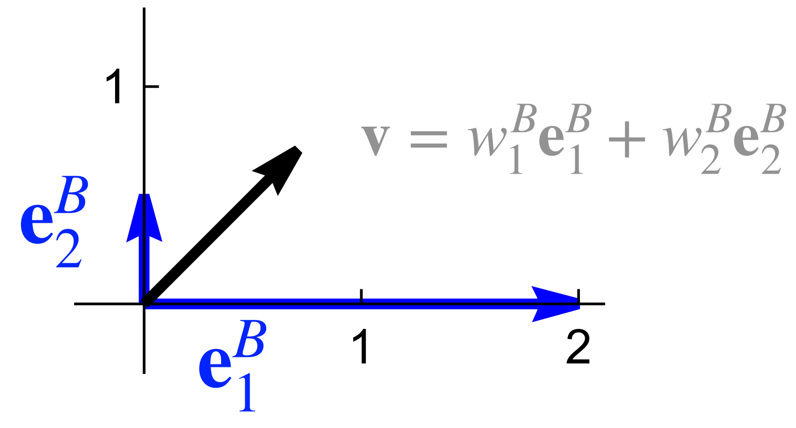
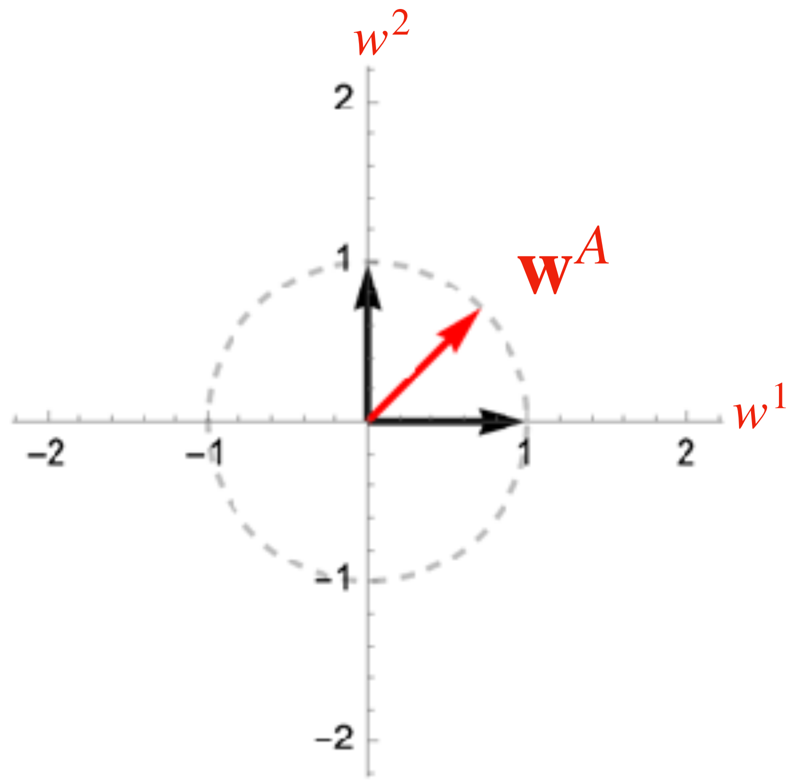


Transforms via $\rho^A(\mathbf{R}_\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Transforms via $\rho^B(\mathbf{R}_\theta) = \begin{pmatrix} \cos\theta & -\frac{1}{4}\sin\theta \\ 4\sin\theta & \cos\theta \end{pmatrix} = Q^{-1}\,\rho^A(\mathbf{R}_\theta)\,Q$



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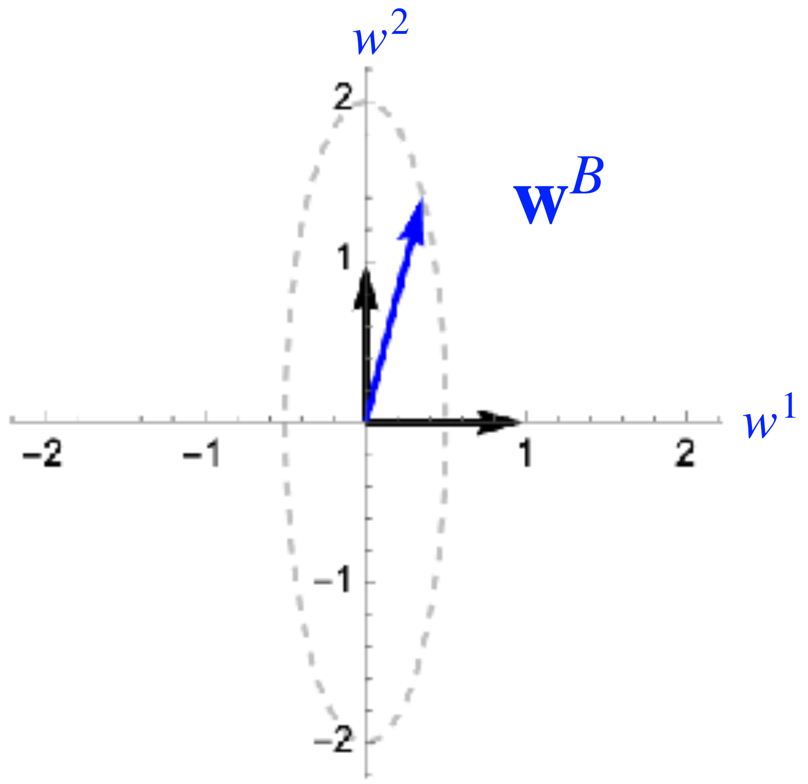

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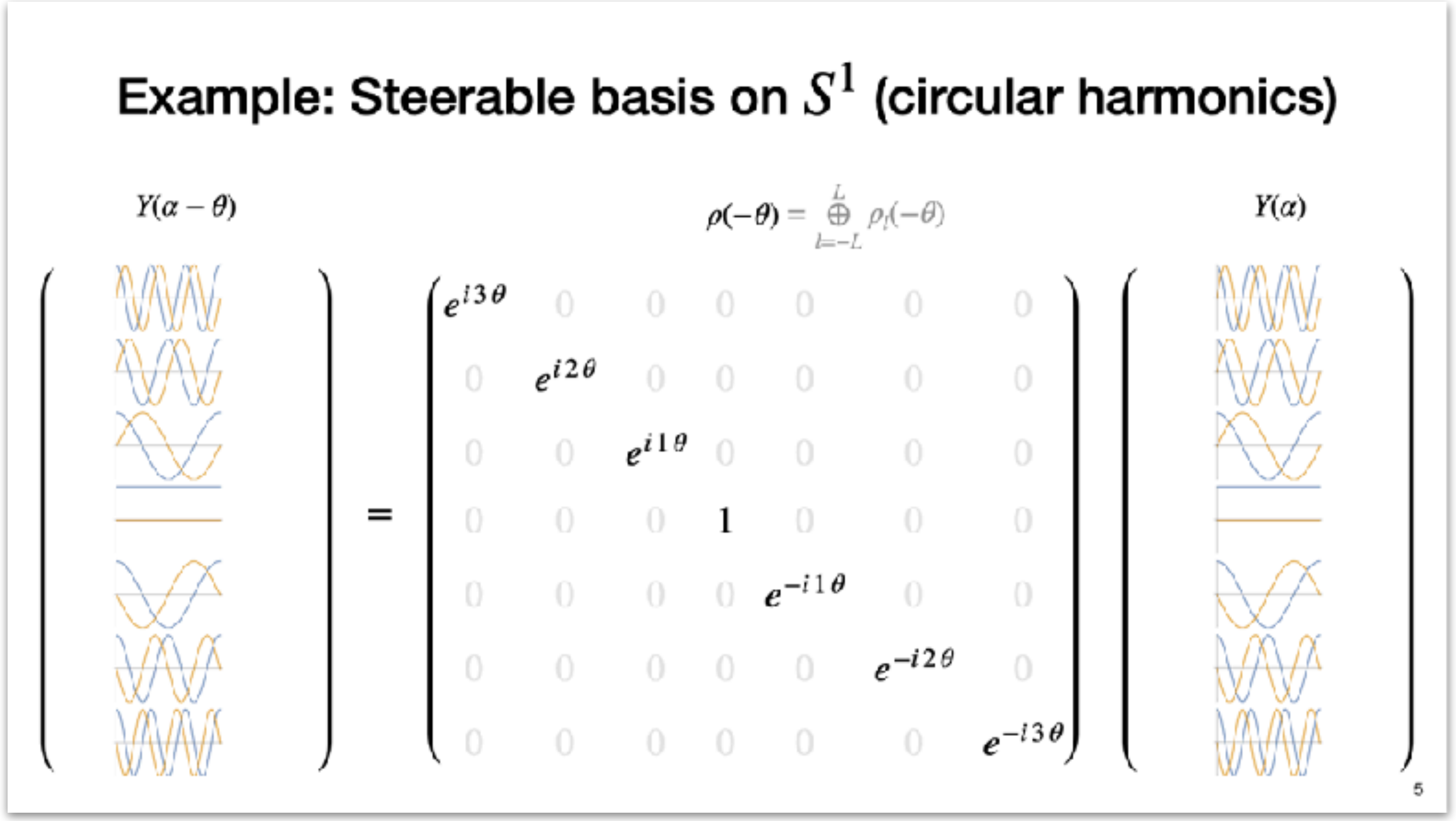

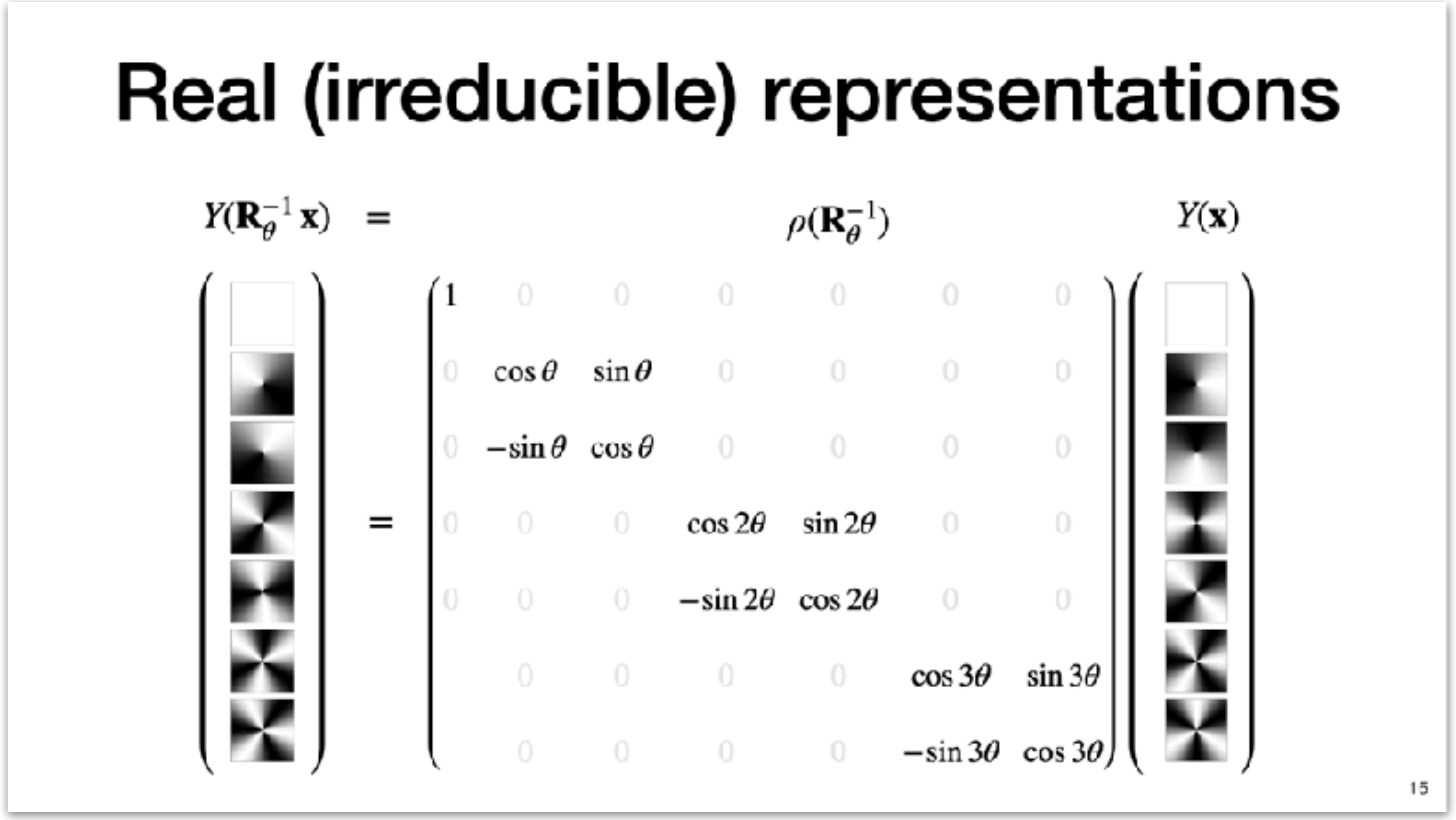

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Represents discrete signal on S^1
(on C_7 in this case)

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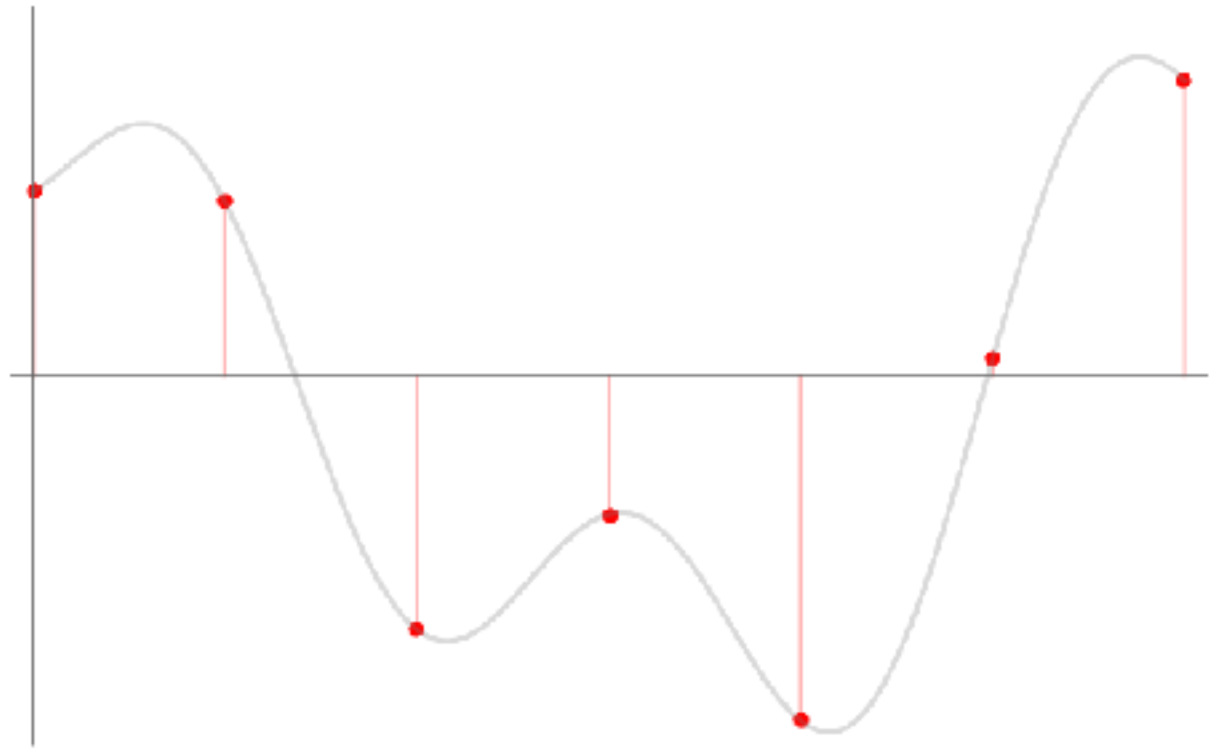

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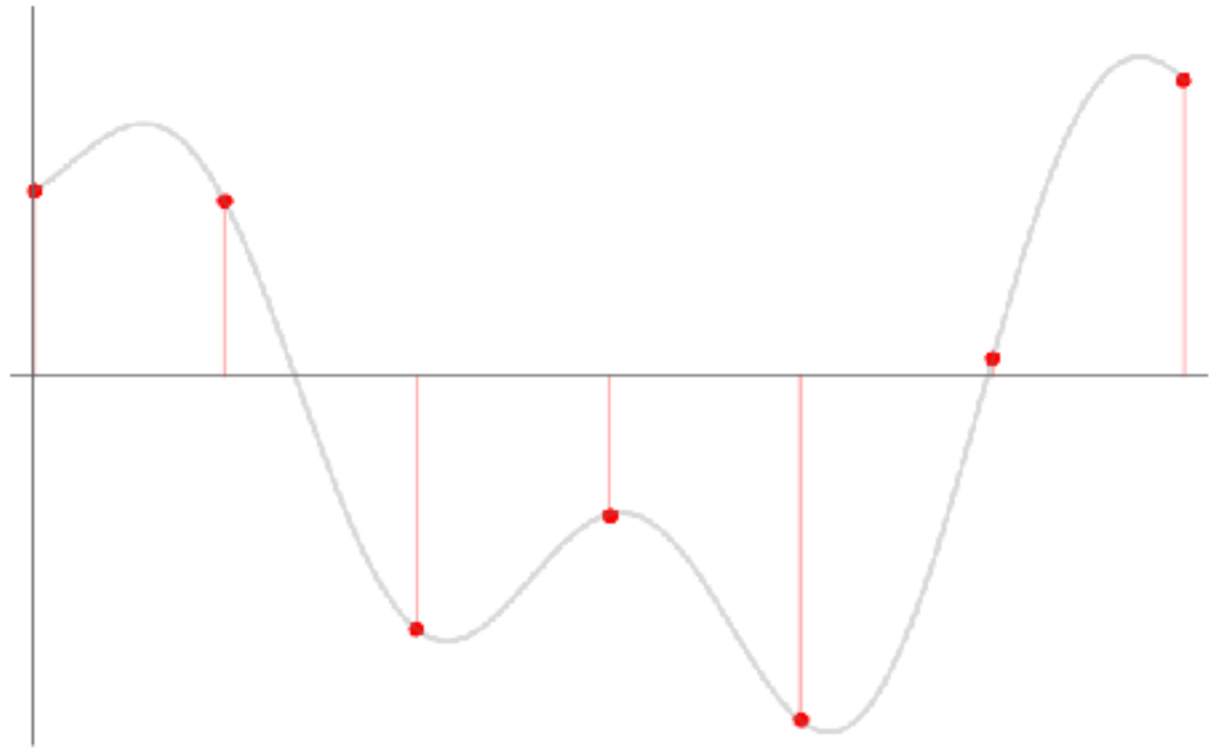

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$$\rho(\theta_n) \qquad \mathbf{v}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} 0.21 \\ 0.20 \\ -0.29 \\ -0.16 \\ -0.39 \\ 0.02 \\ 0.34 \end{pmatrix}$$

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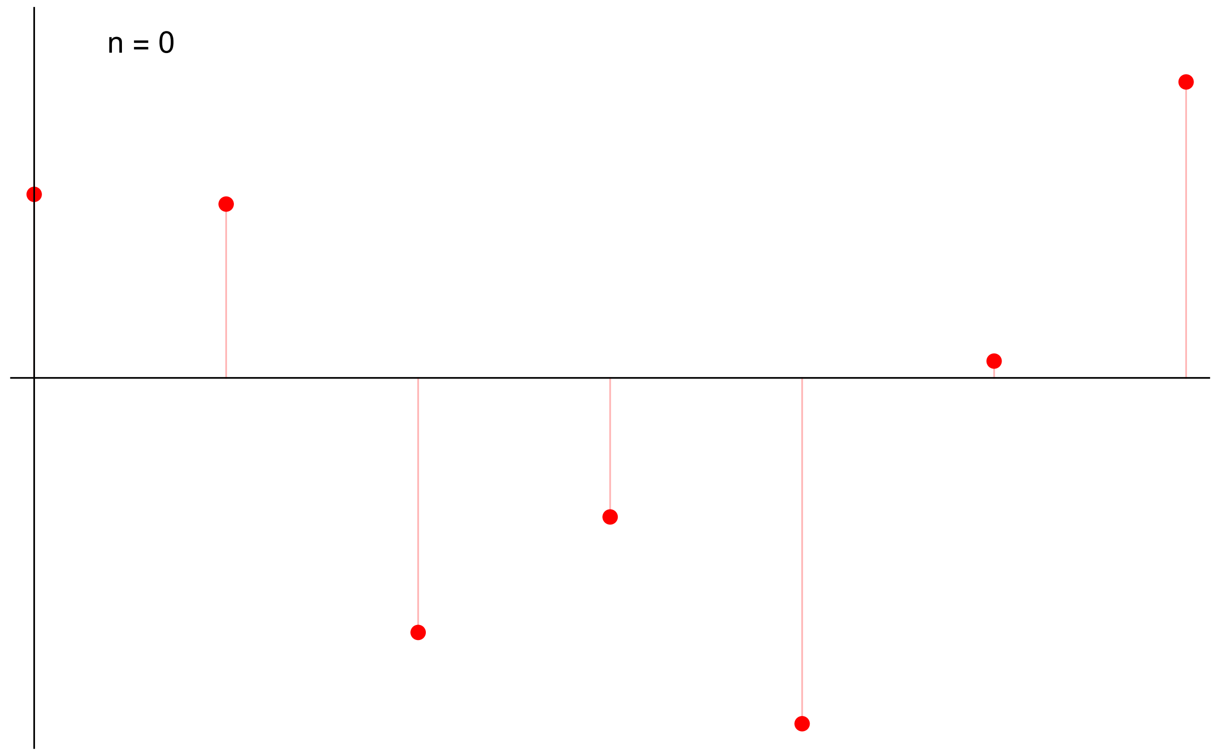

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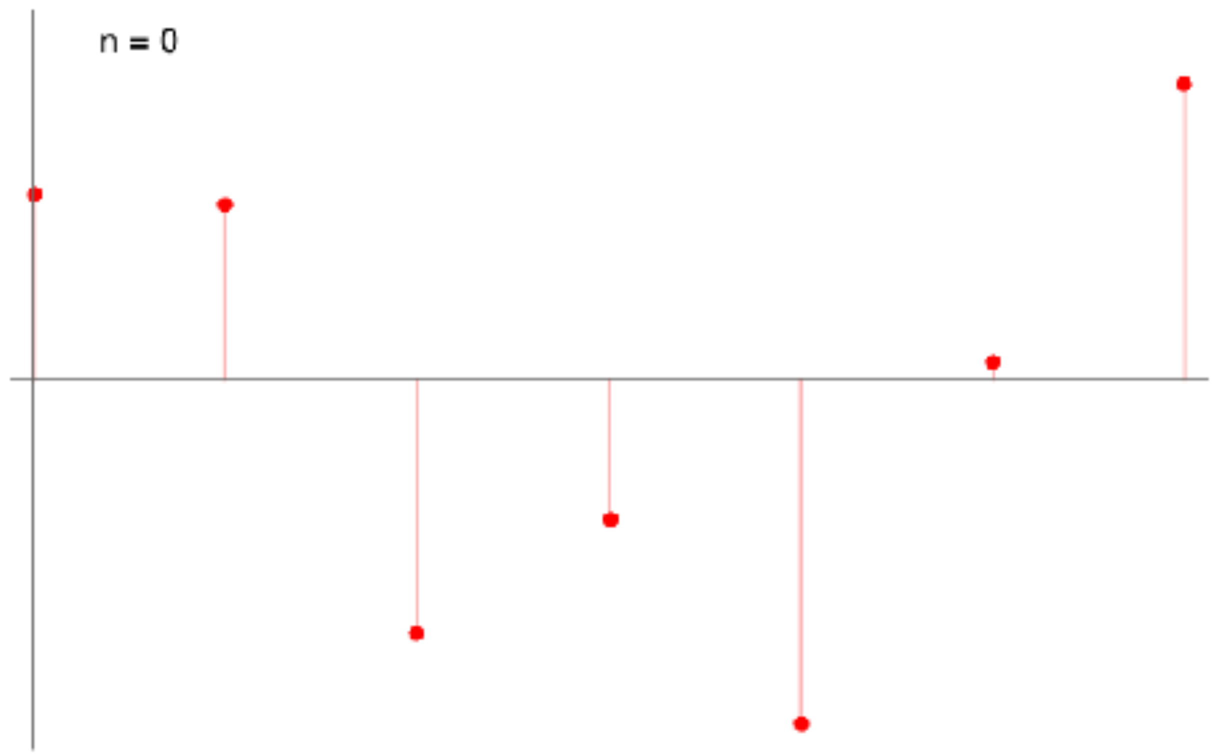

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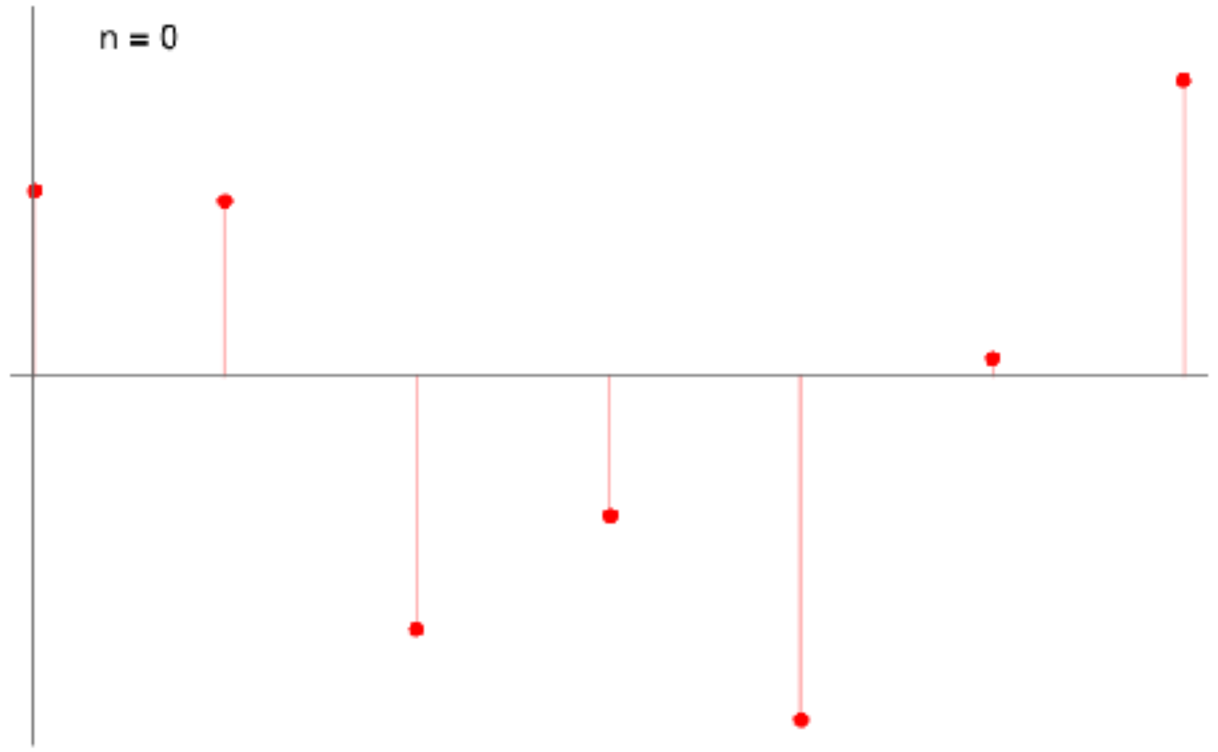

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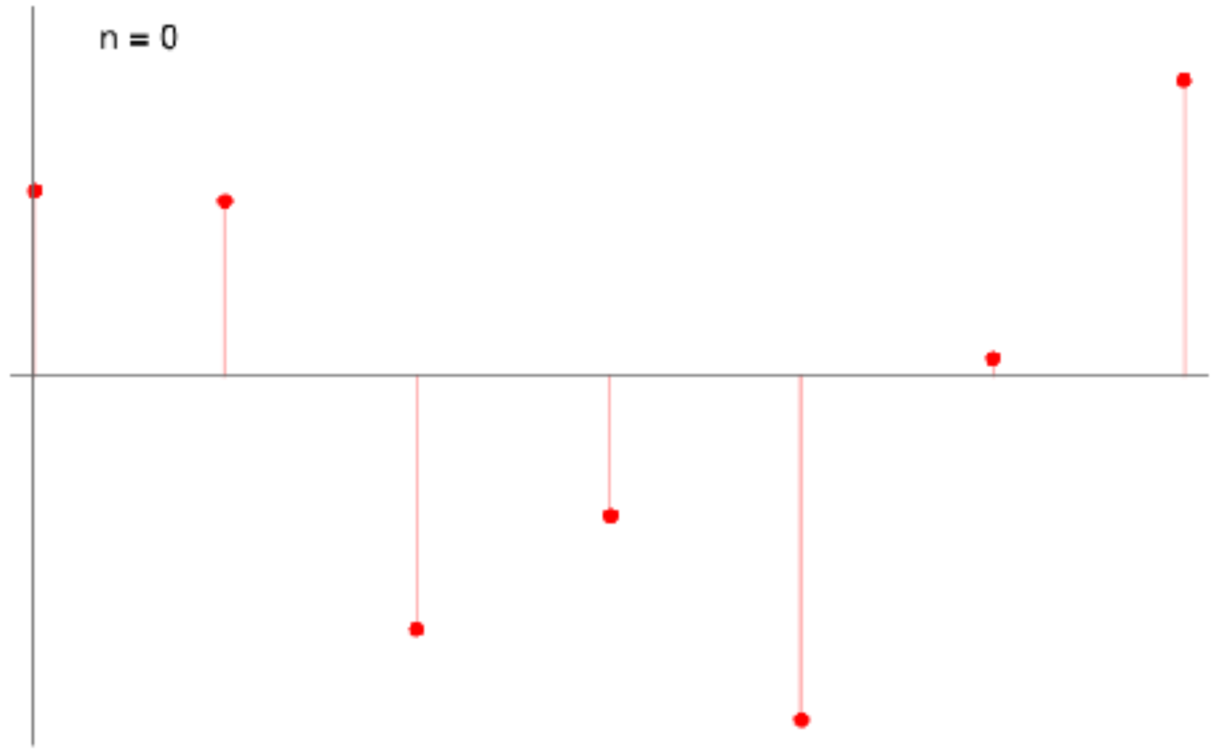

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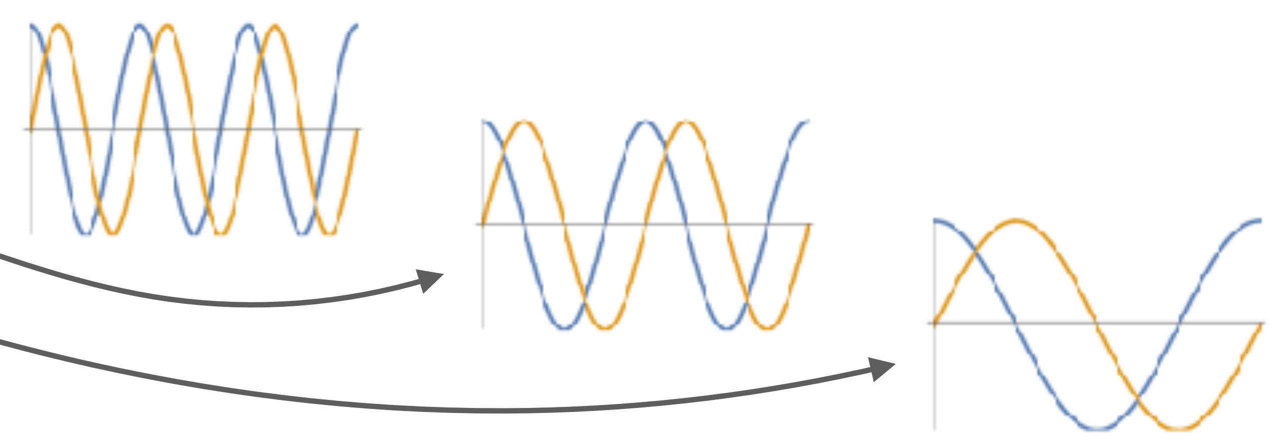


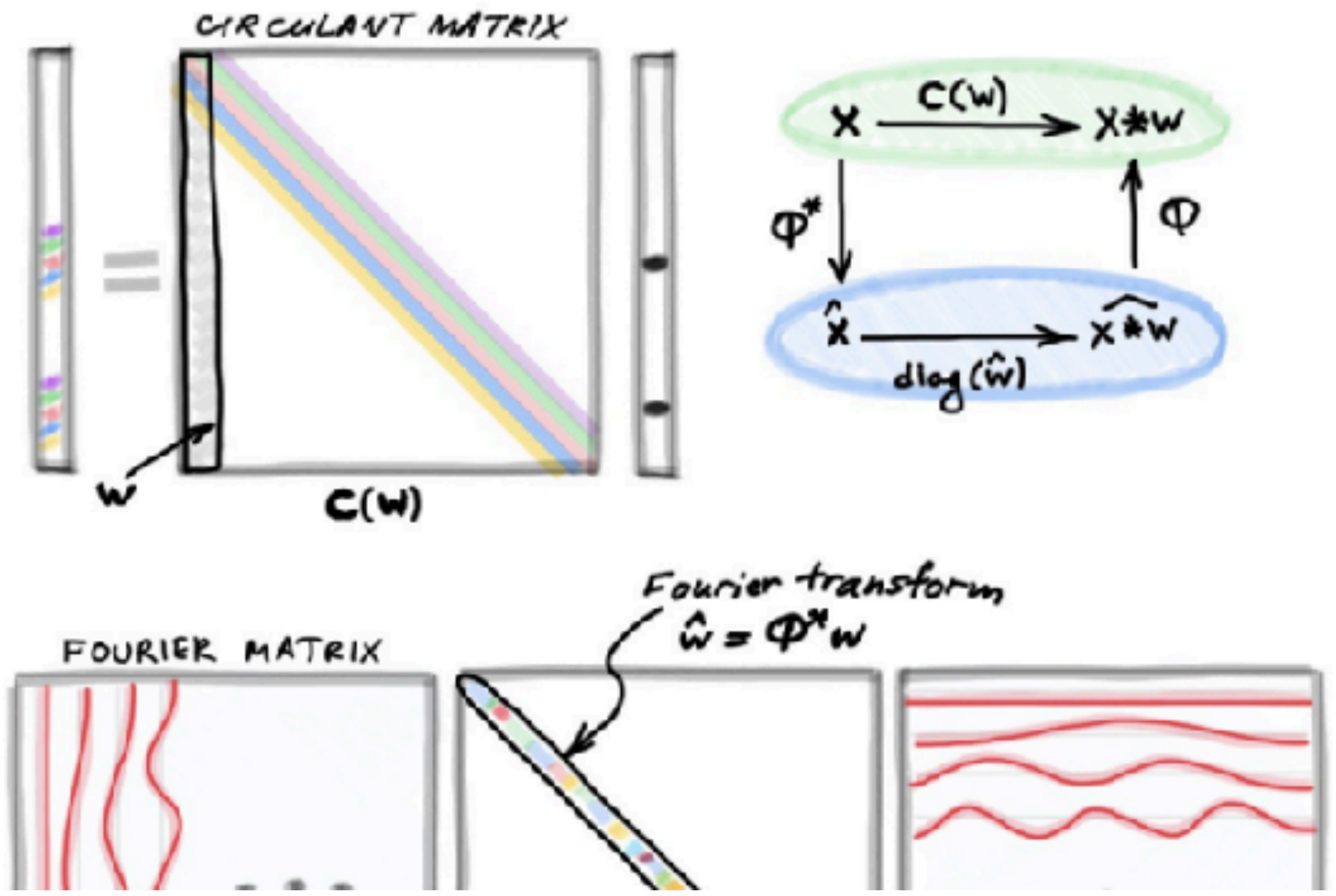
Derive Q (the irreps) yourself through an eigendecomposition of the circular shift matrix

Michael Bronstein
Jul 26, 2020 · 9 min read ★ · Listen

BASICS OF DEEP LEARNING

Deriving convolution from first principles

Have you ever wondered what is so special about convolution? In this post, I derive the convolution from first principles and show that it naturally emerges from translational symmetry.



DISCOVERING TRANSFORMS: A TUTORIAL ON CIRCULANT MATRICES, CIRCULAR CONVOLUTION, AND THE DISCRETE FOURIER TRANSFORM

BASSAM BAMIEH*

4.1. Construction of Eigenvectors/Eigenvalues of S^*. Let w be an eigenvector (with eigenvalue λ) of the shift operator S^*. Note that it is also an eigenvector (with eigenvalue λ^l) of any power $(S^*)^l$ of S^*. Applying the definition (3.3) to the relation $S^*w = \lambda w$ will reveal that an eigenvector w has a very special structure

$$(4.1)\quad \begin{array}{rclcrcll} S^*w &=& \lambda w & \Longleftrightarrow & w_{k+1} &=& \lambda\, w_k, & k \in \mathbb{Z}_n, \\ (S^*)^l w &=& \lambda^l w & \Longleftrightarrow & w_{k+l} &=& \lambda^l\, w_k, & k \in \mathbb{Z}_n,\ l \in \mathbb{Z}, \end{array}$$

i.e. each entry w_{k+1} of w is equal to the previous entry w_k multiplied by the eigenvalue λ. These relations can be used to compute all eigenvectors/eigenvalues of S^*. First, observe that although (4.1) is valid for all $l \in \mathbb{Z}$, this relation "repeats" for $l \geq n$. In particular, for $l = n$ we have for each index k

$$(4.2)\quad w_{k+n} = \lambda^n w_k \quad \Longleftrightarrow \quad w_k = \lambda^n w_k$$

since $k + n \equiv_n k$. Now since the vector $w \neq 0$, then for at least one index k, $w_k \neq 0$, and the last equality implies that $\lambda^n = 1$, i.e. any eigenvalue of S must be an *nth root of unity*

$$\boxed{\lambda^n = 1 \quad \Longleftrightarrow \quad \lambda = \rho_m := e^{i\frac{2\pi}{n}m}, \quad m \in \mathbb{Z}_n.}$$

https://towardsdatascience.com/deriving-convolution-from-first-principles-4ff124888028

https://arxiv.org/pdf/1805.05533.pdf

Overview: Block-diagonalization via Fourier transform

	Finite-dimensional vectors $\mathbf{v} \in \mathbb{R}^d$	**Infinite-dimensional vectors** $f \in \mathbb{L}_2(G)$
Regular representation	$\rho(g_n)\ \mathbf{v}$	$\mathscr{L}_g f$
Decomposed into irreps	$Q^{-1}\left[\bigoplus_{l=-L}^{L} \rho_l(g_n)\right]\ Q$	$\mathscr{F}_G^{-1} \circ \left[\bigoplus_{l=-L}^{L} \rho_l(g)\right] \circ \mathscr{F}_G$
Fourier transform	see below	$[\mathscr{F}_G f]_l = \int_G \rho_l(g) f(g) \mathrm{d}g$
Inverse Fourier transform	$Q^{-1}\ \hat{\mathbf{v}}$	$\mathscr{F}^{-1}[\hat{f}](g) = \sum_l \hat{f}(\rho_l)\rho_l(g^{-1})$

$$Q\ \mathbf{v} = \frac{1}{\sqrt{7}} \begin{pmatrix} 1 & e^{-\frac{6i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{\frac{6i\pi}{7}} \\ 1 & e^{-\frac{4i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{\frac{4i\pi}{7}} \\ 1 & e^{-\frac{2i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{\frac{2i\pi}{7}} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{\frac{2i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{-\frac{2i\pi}{7}} \\ 1 & e^{\frac{4i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{-\frac{4i\pi}{7}} \\ 1 & e^{\frac{6i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{-\frac{6i\pi}{7}} \end{pmatrix} \begin{pmatrix} 0.21 \\ 0.20 \\ -0.29 \\ -0.16 \\ -0.39 \\ 0.02 \\ 0.34 \end{pmatrix}$$

$$= \sum_l d_{\rho_l} \operatorname{tr}\left[\hat{f}(\rho_l)\rho_l(h^{-1})\right]$$

General case** $d_l \times d_l$ **matrix irreps of compact groups

See e.g. Kondor, R., & Trivedi, S. (2018, July). On the generalization of equivariance and convolution in neural networks to the action of compact groups. In International Conference on Machine Learning (pp. 2747-2755). PMLR.

Overview: Block-diagonalization via Fourier transform

	Finite-dimensional vectors $\mathbf{v} \in \mathbb{R}^d$	**Infinite-dimensional vectors** $f \in \mathbb{L}_2(G)$
Regular representation	$\rho(g_n)\ \mathbf{v}$	$\mathscr{L}_g f$
Decomposed into irreps	$Q^{-1}\left[\bigoplus_{l=-L}^{L} \rho_l(g_n)\right] Q$	$\mathscr{F}_G^{-1} \circ \left[\bigoplus_{l=-L}^{L} \rho_l(g)\right] \circ \mathscr{F}_G$
Fourier transform	$Q\ \mathbf{v} = \frac{1}{\sqrt{7}} \begin{pmatrix} 1 & e^{-\frac{6i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{\frac{6i\pi}{7}} \\ 1 & e^{-\frac{4i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{\frac{4i\pi}{7}} \\ 1 & e^{-\frac{2i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{\frac{2i\pi}{7}} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{\frac{2i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{-\frac{2i\pi}{7}} \\ 1 & e^{\frac{4i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{-\frac{4i\pi}{7}} \\ 1 & e^{\frac{6i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{-\frac{6i\pi}{7}} \end{pmatrix} \begin{pmatrix} 0.21 \\ 0.20 \\ -0.29 \\ -0.16 \\ -0.39 \\ 0.02 \\ 0.34 \end{pmatrix}$	$[\mathscr{F}_G f]_l = \int_G \rho_l(g) f(g) \mathrm{d}g$
Inverse Fourier transform	$Q^{-1}\ \hat{\mathbf{v}}$	$\mathscr{F}^{-1}[\hat{f}](g) = \sum_l \hat{f}(\rho_l)\rho_l(g^{-1})$

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Fourier shift theorem!!!

General case $d_l \times d_l$ matrix irreps of compact groups

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Fourier shift theorem!!!

Fourier transform (finite-dimensional):

$$Q\ \mathbf{v} = \frac{1}{\sqrt{7}}\begin{pmatrix} 1 & e^{-\frac{6i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{\frac{6i\pi}{7}} \\ 1 & e^{-\frac{4i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{\frac{4i\pi}{7}} \\ 1 & e^{-\frac{2i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{\frac{2i\pi}{7}} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{\frac{2i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{-\frac{2i\pi}{7}} \\ 1 & e^{\frac{4i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{-\frac{4i\pi}{7}} \\ 1 & e^{\frac{6i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{-\frac{6i\pi}{7}} \end{pmatrix}\begin{pmatrix} 0.21 \\ 0.20 \\ -0.29 \\ -0.16 \\ -0.39 \\ 0.02 \\ 0.34 \end{pmatrix}$$

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