UNIVERSITY OF AMSTERDAM

AMLAB Amsterdam Machine Learning Lab

Group Equivariant Deep Learning

Lecture 2 - Steerable group convolutions

Lecture 2.2 - Revisiting regular G-convs with steerable kernels

Motivating the Fourier transform on H and showing we now no longer need a grid on the sub-group H!

Erik Bekkers, Amsterdam Machine Learning Lab, University of Amsterdam

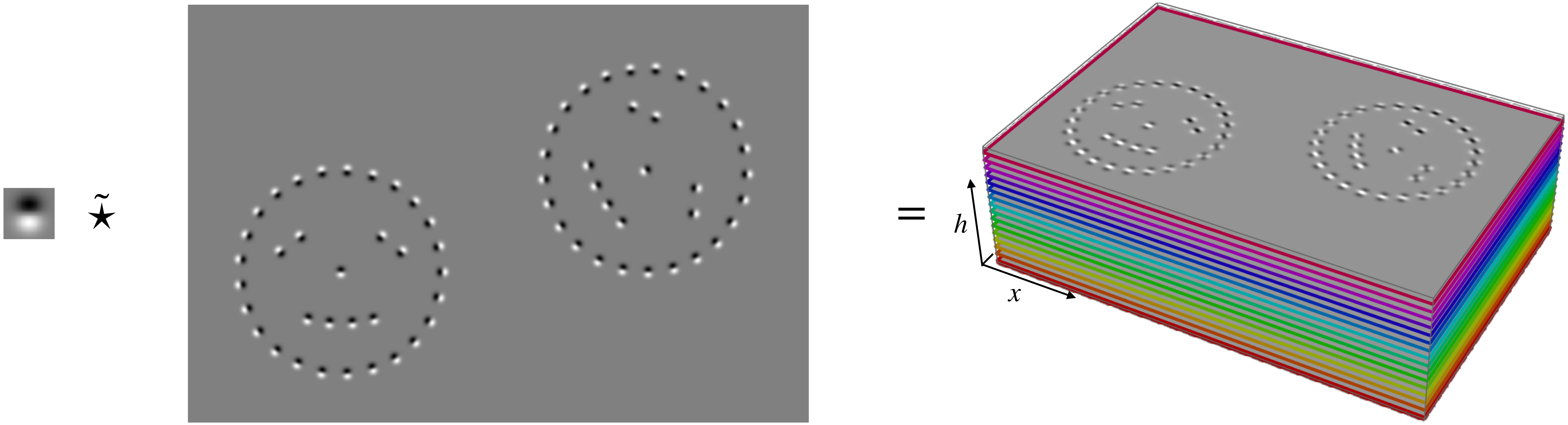
This mini-course serves as a module with the UvA Master AI course Deep Learning 2 https://uvadl2c.github.io/

Regular group convolutions revisited

Group convolution ($G = \mathbb{R}^d \rtimes H$):

$$(k \,\tilde{\star}\, f)(g) = (\mathscr{L}_g^{G \to \mathbb{L}_2(X)}\, k \,,\, f)_{\mathbb{L}_2(X)}$$

(e.g. $G = SE(2) = \mathbb{R}^2 \rtimes SO(2)$)
$X = \mathbb{R}^2$



2D convolution kernel 2D input feature map *SE(2)* output feature map

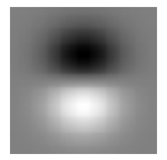
Regular group convolutions revisited

Group convolution ($G = \mathbb{R}^d \rtimes H$):

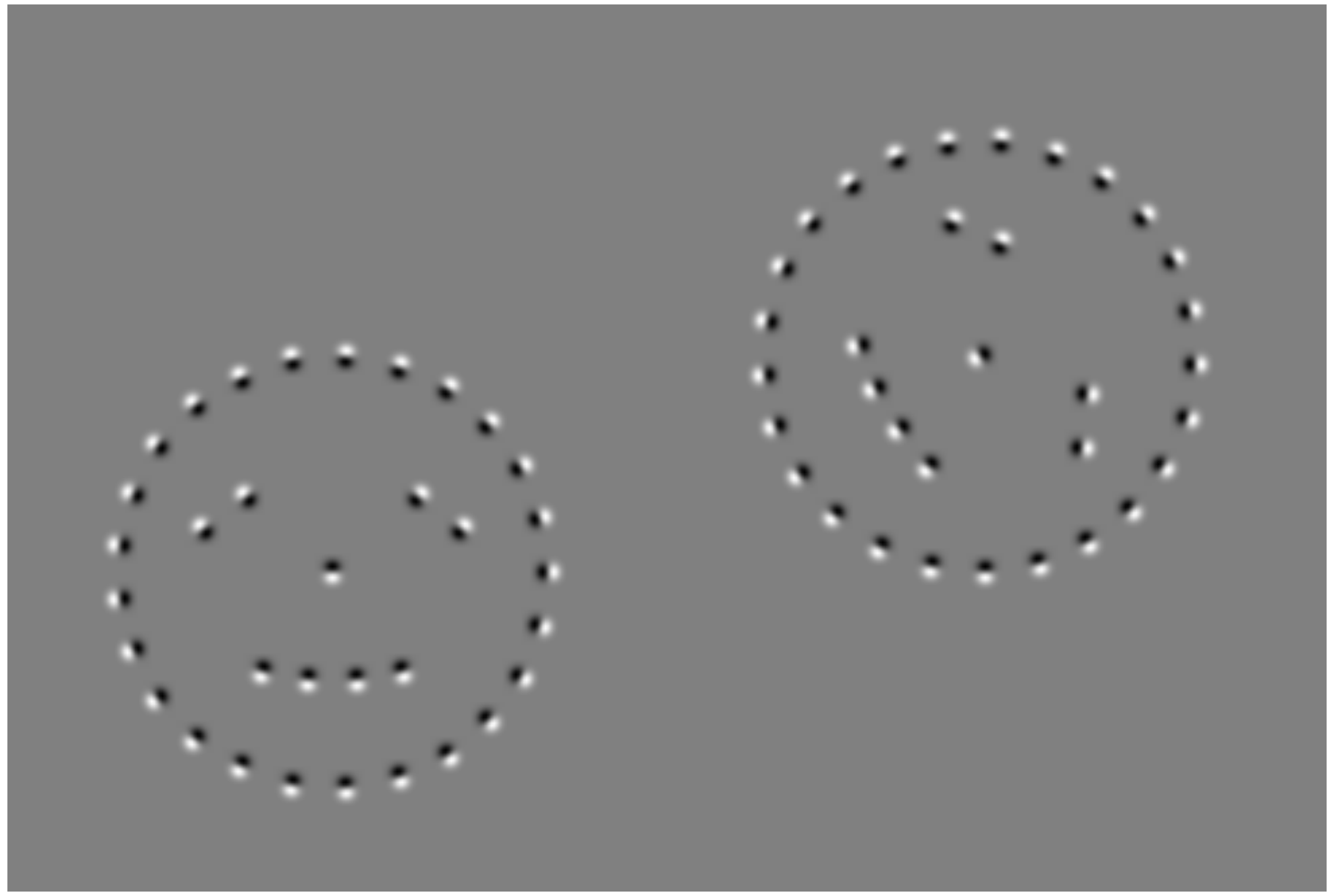
(e.g. $G = SE(2) = \mathbb{R}^2 \rtimes SO(2)$)
$X = \mathbb{R}^2$

$$(k \,\tilde{\star}\, f)(g) = (\mathscr{L}_g^{G \to \mathbb{L}_2(X)}\, k\,,\, f)_{\mathbb{L}_2(X)}$$

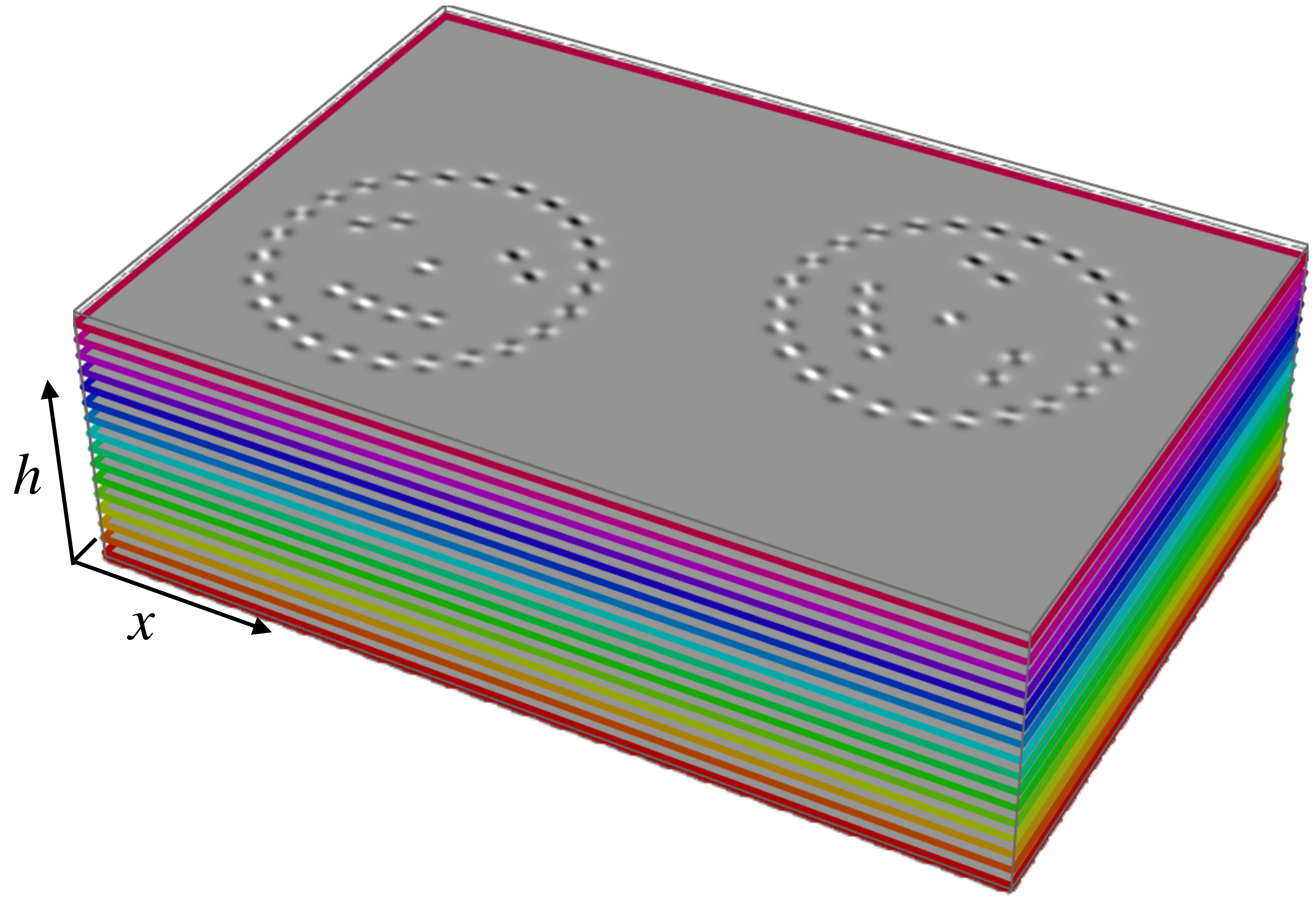
$$= \int_{\mathbb{R}^d} k(g^{-1}\mathbf{x}')f(\mathbf{x}')d\mathbf{x}'$$



$\tilde{\star}$



$=$



2D convolution kernel

2D input feature map

SE(2) output feature map

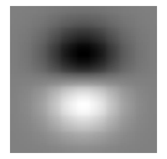
Regular group convolutions revisited

Group convolution ($G = \mathbb{R}^d \rtimes H$):

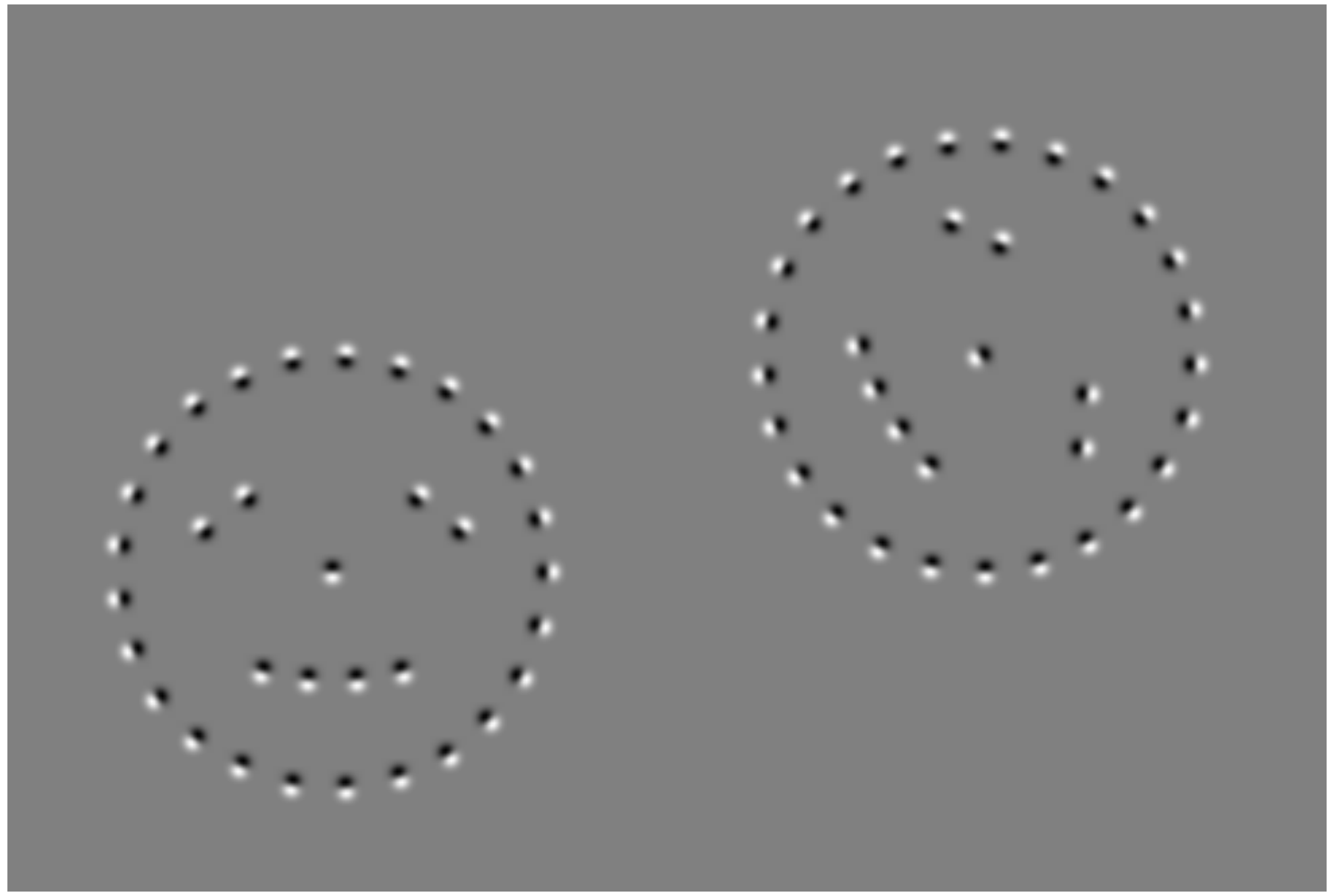
(e.g. $G = SE(2) = \mathbb{R}^2 \rtimes SO(2)$)
$X = \mathbb{R}^2$

$$(k \tilde{\star} f)(g) = (\mathscr{L}_g^{G \to \mathbb{L}_2(X)} k \, , f)_{\mathbb{L}_2(X)} \quad = (\mathscr{L}_{\mathbf{x}}^{(\mathbf{R}^d,+) \to \mathbb{L}_2(\mathbb{R}^d)} \mathscr{L}_h^{H \to \mathbb{L}_2(\mathbb{R}^d)} k \, , f)_{\mathbb{L}_2(\mathbb{R}^d)}$$

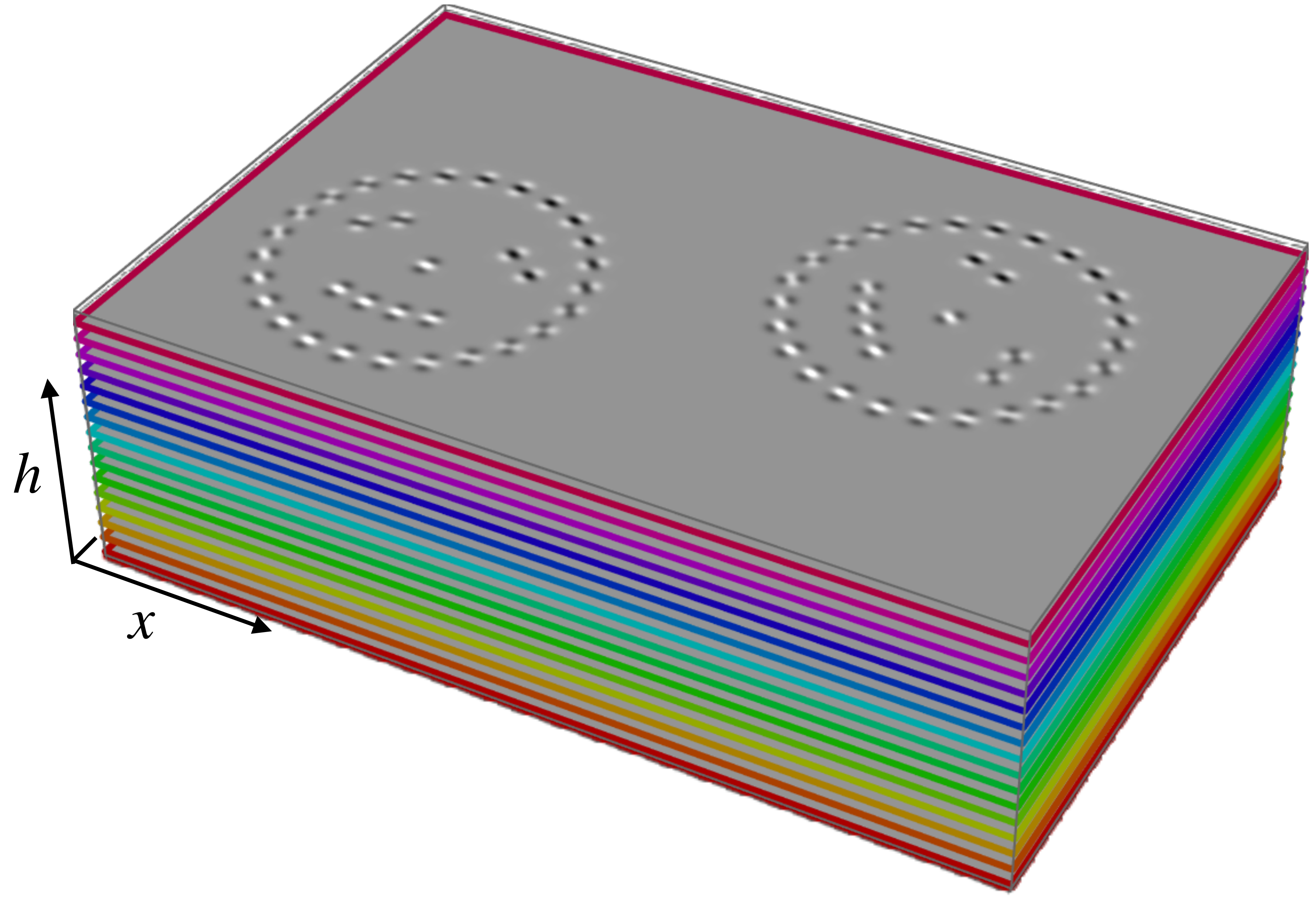
$$= \int_{\mathbb{R}^d} k(g^{-1}\mathbf{x}') f(\mathbf{x}') \mathrm{d}\mathbf{x}'$$



$\tilde{\star}$



$=$



2D convolution kernel

2D input feature map

SE(2) output feature map

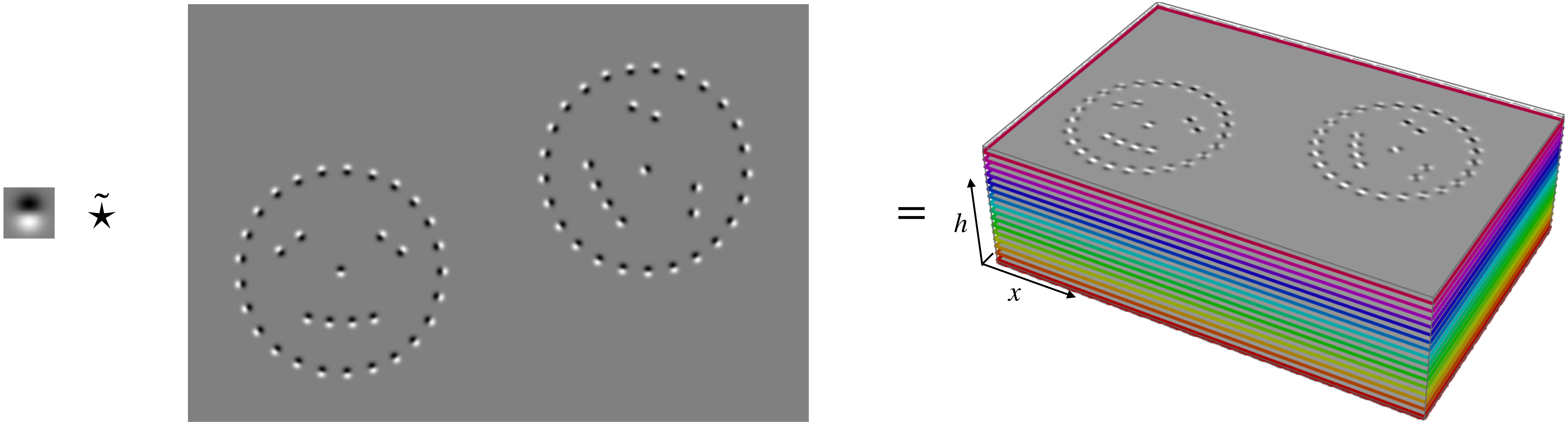
Regular group convolutions revisited

Group convolution ($G = \mathbb{R}^d \rtimes H$):

(e.g. $G = SE(2) = \mathbb{R}^2 \rtimes SO(2)$)
$X = \mathbb{R}^2$

$$(k \,\tilde{\star}\, f)(g) = (\mathscr{L}_g^{G\to\mathbb{L}_2(X)}\, k \,,\, f)_{\mathbb{L}_2(X)} \quad = (\mathscr{L}_{\mathbf{x}}^{(\mathbf{R}^d,+)\to\mathbb{L}_2(\mathbb{R}^d)} \mathscr{L}_h^{H\to\mathbb{L}_2(\mathbb{R}^d)}\, k \,,\, f)_{\mathbb{L}_2(\mathbb{R}^d)}$$

$$= \int_{\mathbb{R}^d} k(g^{-1}\mathbf{x}')f(\mathbf{x}')\mathrm{d}\mathbf{x}' \quad = \int_{\mathbb{R}^2} k(\mathbf{R}_\theta^{-1}(\mathbf{x}'-\mathbf{x}))f(\mathbf{x}')\mathrm{d}\mathbf{x}'$$



2D convolution kernel 2D input feature map *SE(2)* output feature map

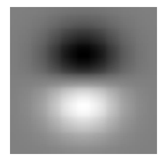
Regular group convolutions revisited

Group convolution ($G = \mathbb{R}^d \rtimes H$):

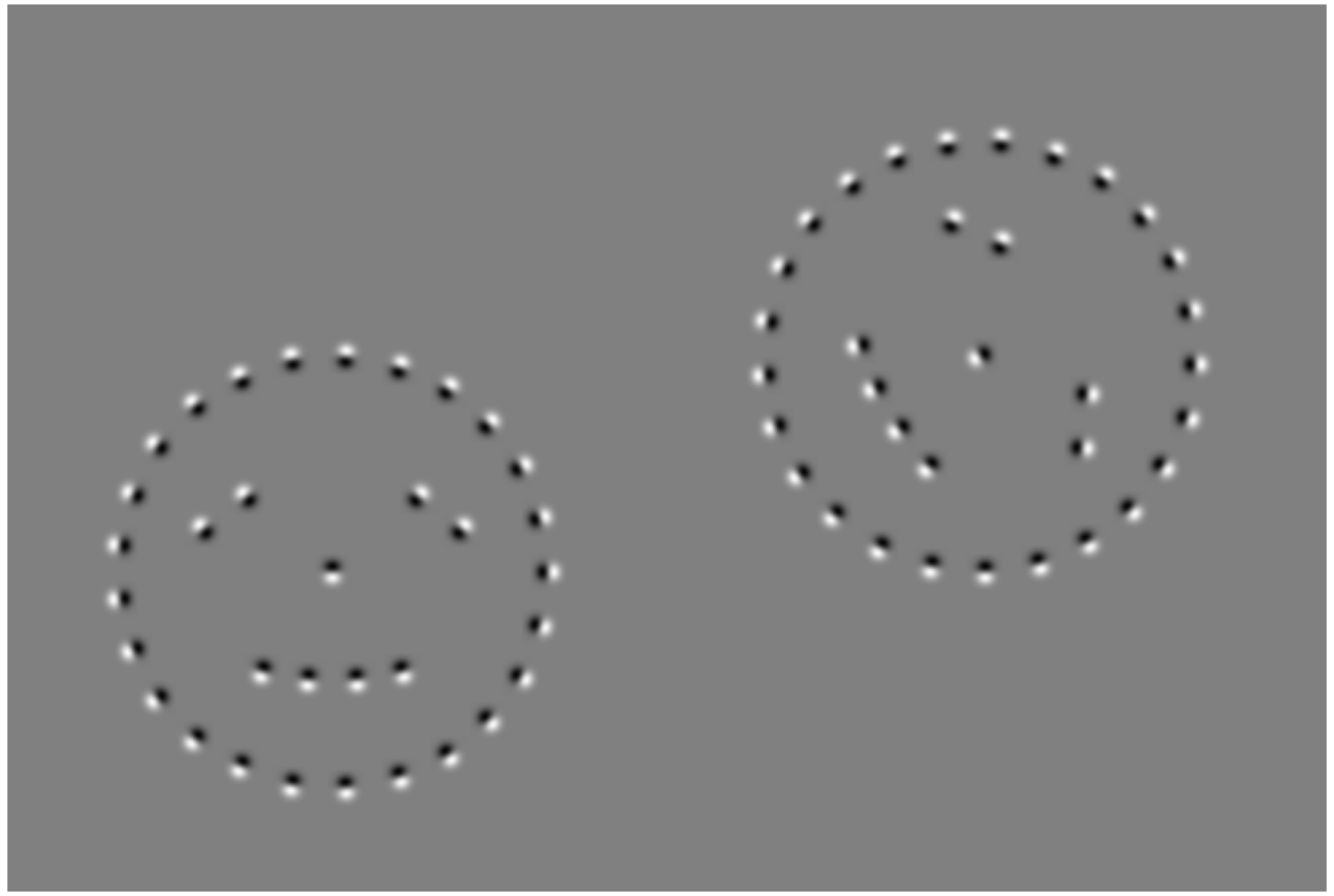
(e.g. $G = SE(2) = \mathbb{R}^2 \rtimes SO(2)$)
$X = \mathbb{R}^2$

$$(k \,\tilde{\star}\, f)(g) = (\mathscr{L}_g^{G\to\mathbb{L}_2(X)}\, k \,,\, f)_{\mathbb{L}_2(X)} = (\mathscr{L}_{\mathbf{x}}^{(\mathbf{R}^d,+)\to\mathbb{L}_2(\mathbb{R}^d)}\, \mathscr{L}_h^{H\to\mathbb{L}_2(\mathbb{R}^d)}\, k \,,\, f)_{\mathbb{L}_2(\mathbb{R}^d)}$$

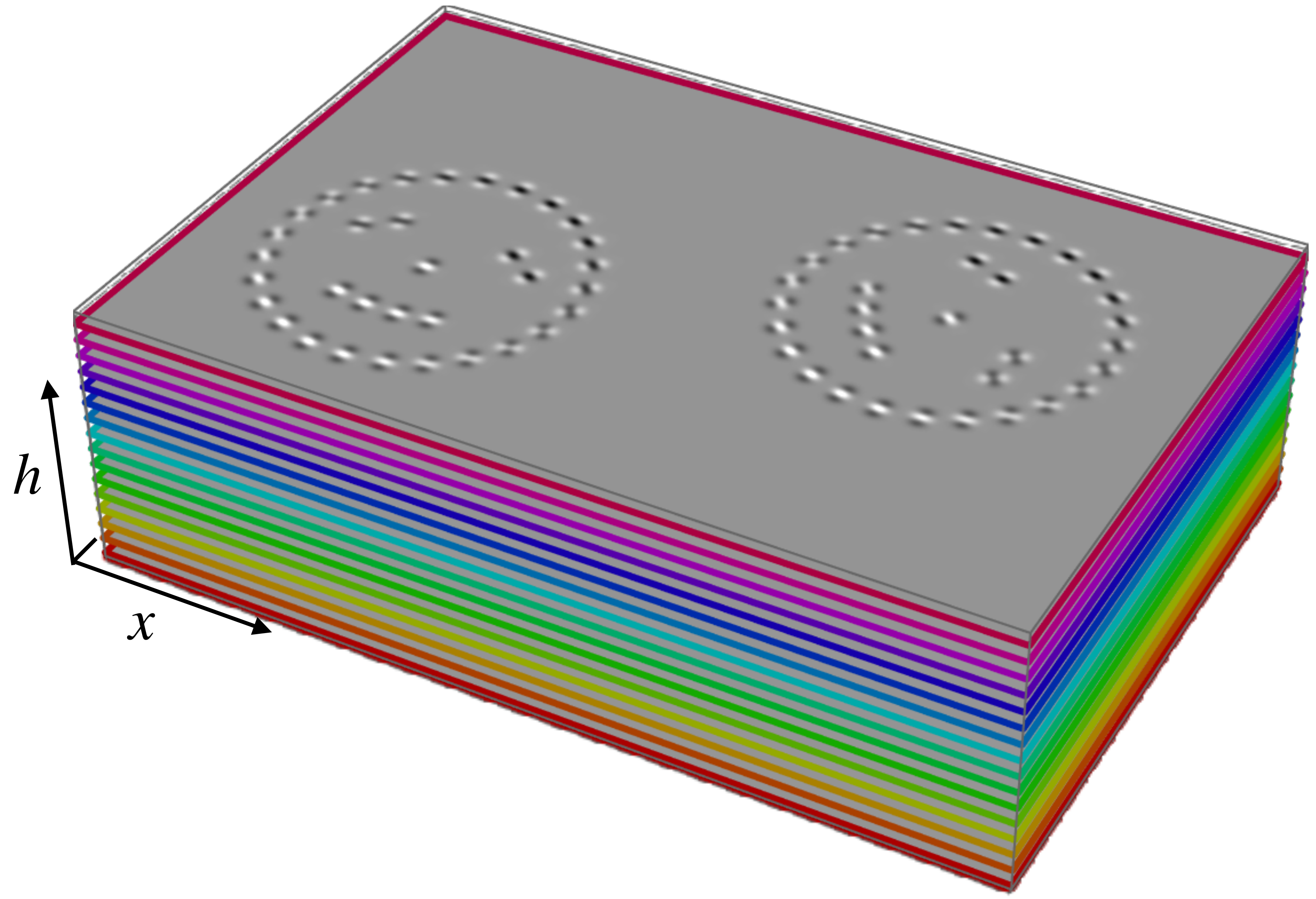
$$= \int_{\mathbb{R}^d} k(g^{-1}\mathbf{x}') f(\mathbf{x}') \mathrm{d}\mathbf{x}' \quad = \int_{\mathbb{R}^2} k_\theta(\mathbf{x}' - \mathbf{x}) f(\mathbf{x}') \mathrm{d}\mathbf{x}'$$



$\tilde{\star}$



$=$



2D convolution kernel

2D input feature map

SE(2) output feature map

Regular group convolutions revisited

Group convolution ($G = \mathbb{R}^d \rtimes H$):

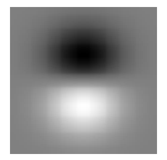
(e.g. $G = SE(2) = \mathbb{R}^2 \rtimes SO(2)$)
$X = \mathbb{R}^2$

$$(k \,\tilde{\star}\, f)(g) = (\mathscr{L}_g^{G\to\mathbb{L}_2(X)} k \,,\, f)_{\mathbb{L}_2(X)} = (\mathscr{L}_{\mathbf{x}}^{(\mathbb{R}^d,+)\to\mathbb{L}_2(\mathbb{R}^d)} \mathscr{L}_h^{H\to\mathbb{L}_2(\mathbb{R}^d)} k \,,\, f)_{\mathbb{L}_2(\mathbb{R}^d)}$$

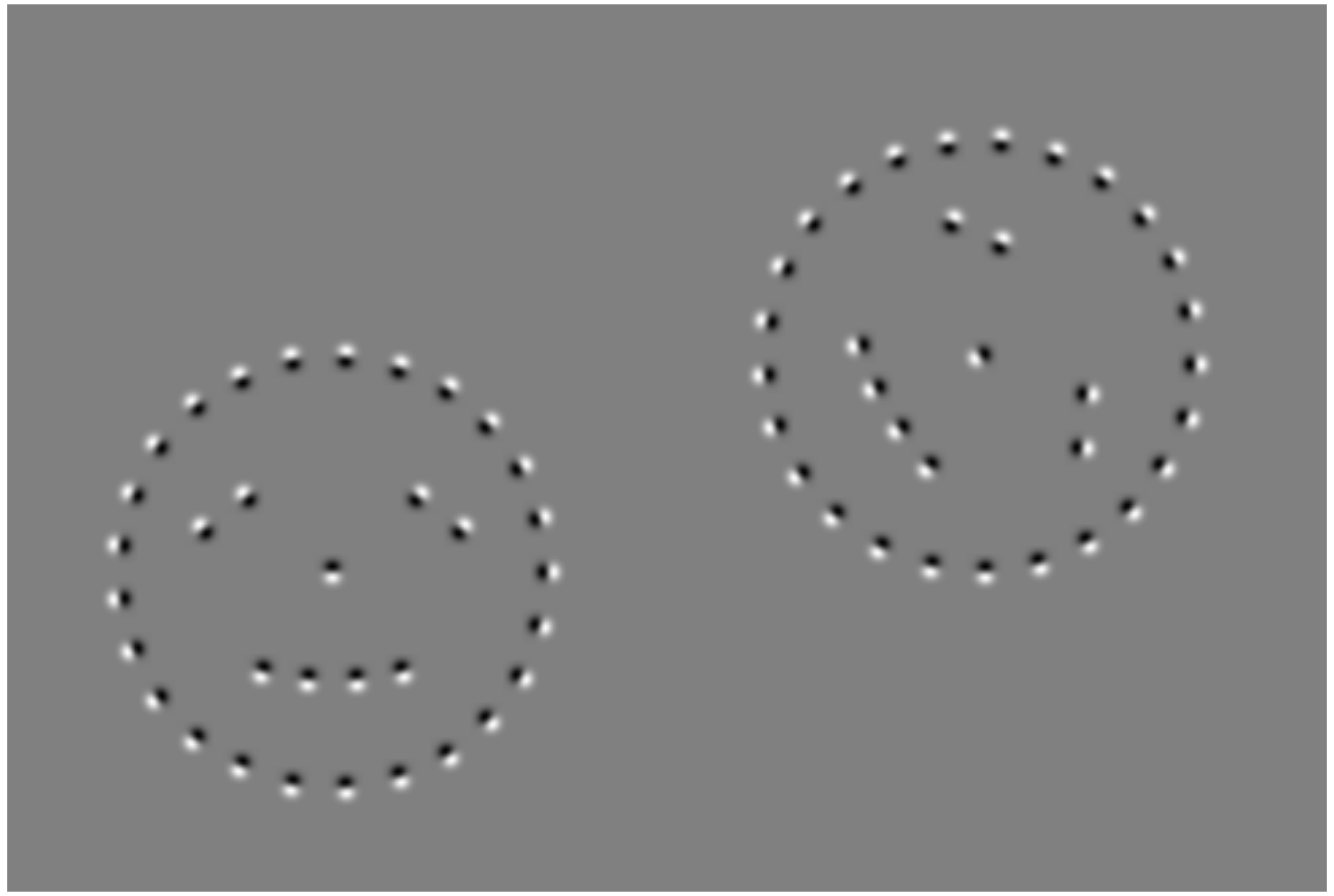
$$= \int_{\mathbb{R}^d} k(g^{-1}\mathbf{x}') f(\mathbf{x}') \mathrm{d}\mathbf{x}' = \int_{\mathbb{R}^2} k_\theta(\mathbf{x}' - \mathbf{x}) f(\mathbf{x}') \mathrm{d}\mathbf{x}'$$

---

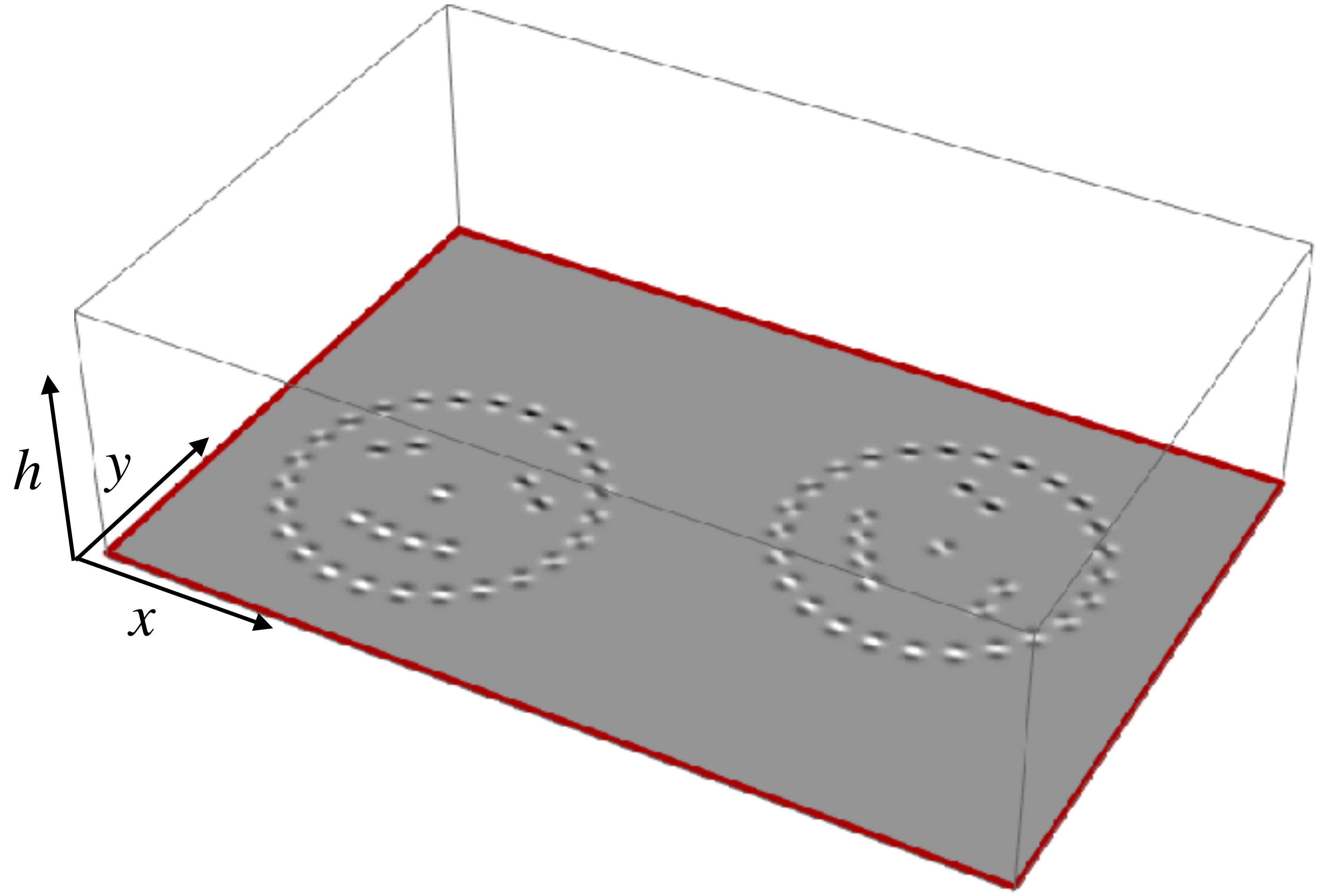
translation
“template matching”



$\star_{\mathbb{R}^2}$



=



Rotated
2D convolution kernel

2D input feature map

SE(2) output feature map

Regular group convolutions revisited

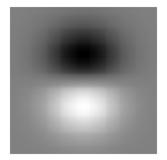
Group convolution ($G = \mathbb{R}^d \rtimes H$):

(e.g. $G = SE(2) = \mathbb{R}^2 \rtimes SO(2)$)
$X = \mathbb{R}^2$

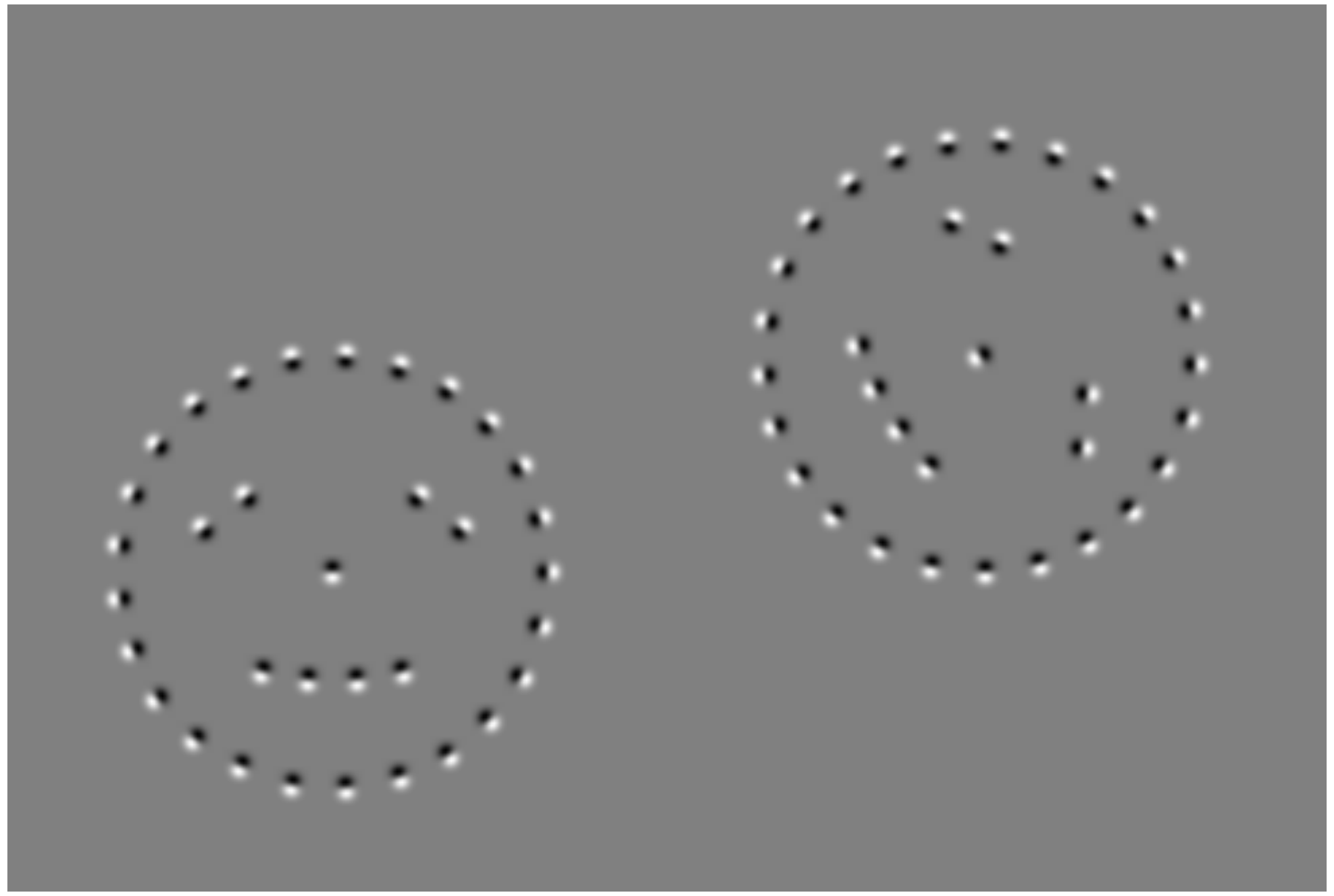
$$(k \tilde{\star} f)(g) = (\mathscr{L}_g^{G\to\mathbb{L}_2(X)}\, k\,,\, f)_{\mathbb{L}_2(X)} = (\mathscr{L}_{\mathbf{x}}^{(\mathbb{R}^d,+)\to\mathbb{L}_2(\mathbb{R}^d)}\, \mathscr{L}_h^{H\to\mathbb{L}_2(\mathbb{R}^d)}\, k\,,\, f)_{\mathbb{L}_2(\mathbb{R}^d)}$$

$$= \int_{\mathbb{R}^d} k(g^{-1}\mathbf{x}')f(\mathbf{x}')d\mathbf{x}' = \int_{\mathbb{R}^2} k_\theta(\mathbf{x}'-\mathbf{x})f(\mathbf{x}')d\mathbf{x}'$$

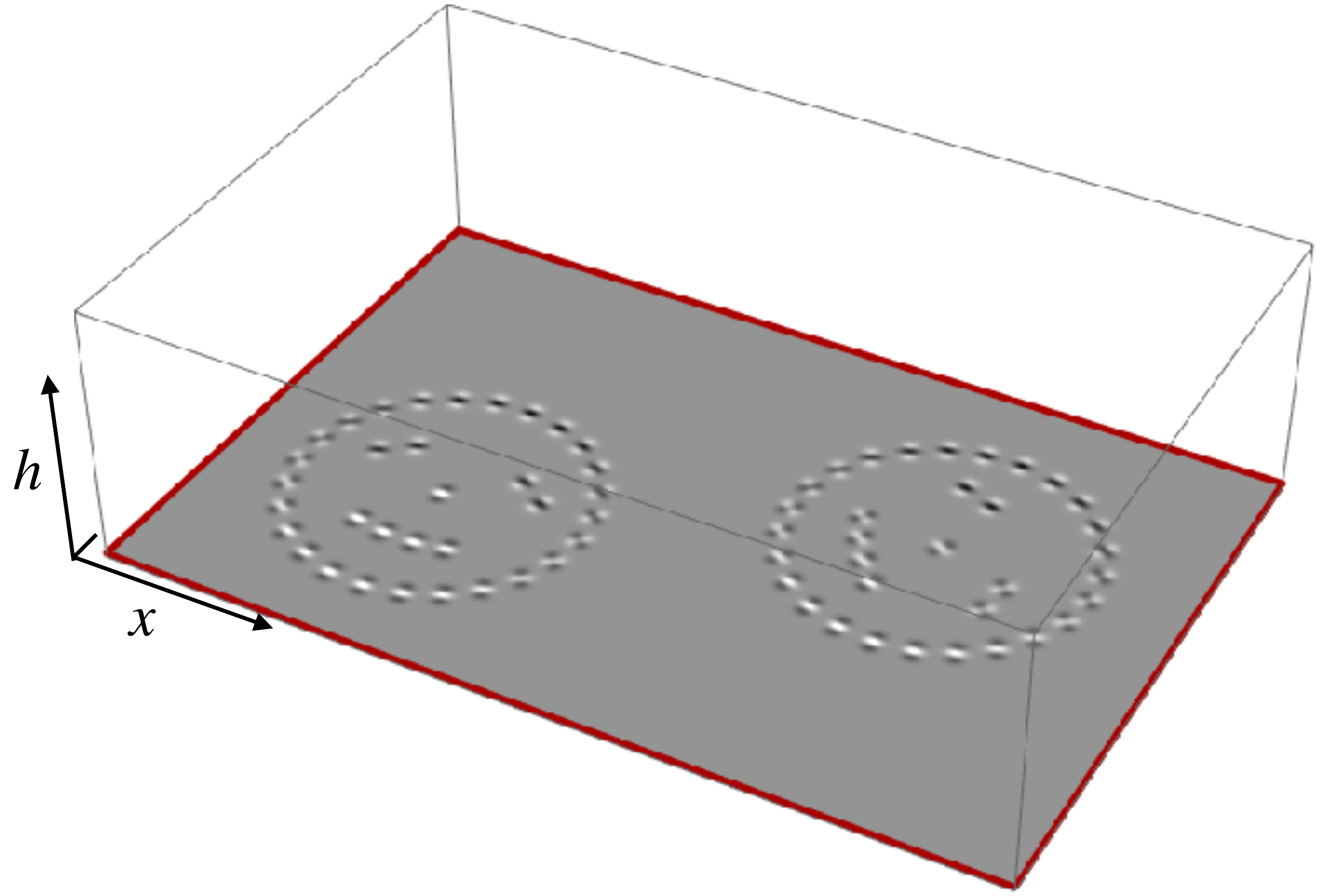
translation
"template matching"



$\star_{\mathbb{R}^2}$



=



Rotated
2D convolution kernel

2D input feature map

SE(2) output feature map

Regular group convolutions revisited

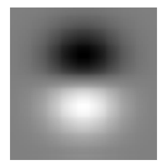
Group convolution ($G = \mathbb{R}^d \rtimes H$):

(e.g. $G = SE(2) = \mathbb{R}^2 \rtimes SO(2)$)
$X = \mathbb{R}^2$

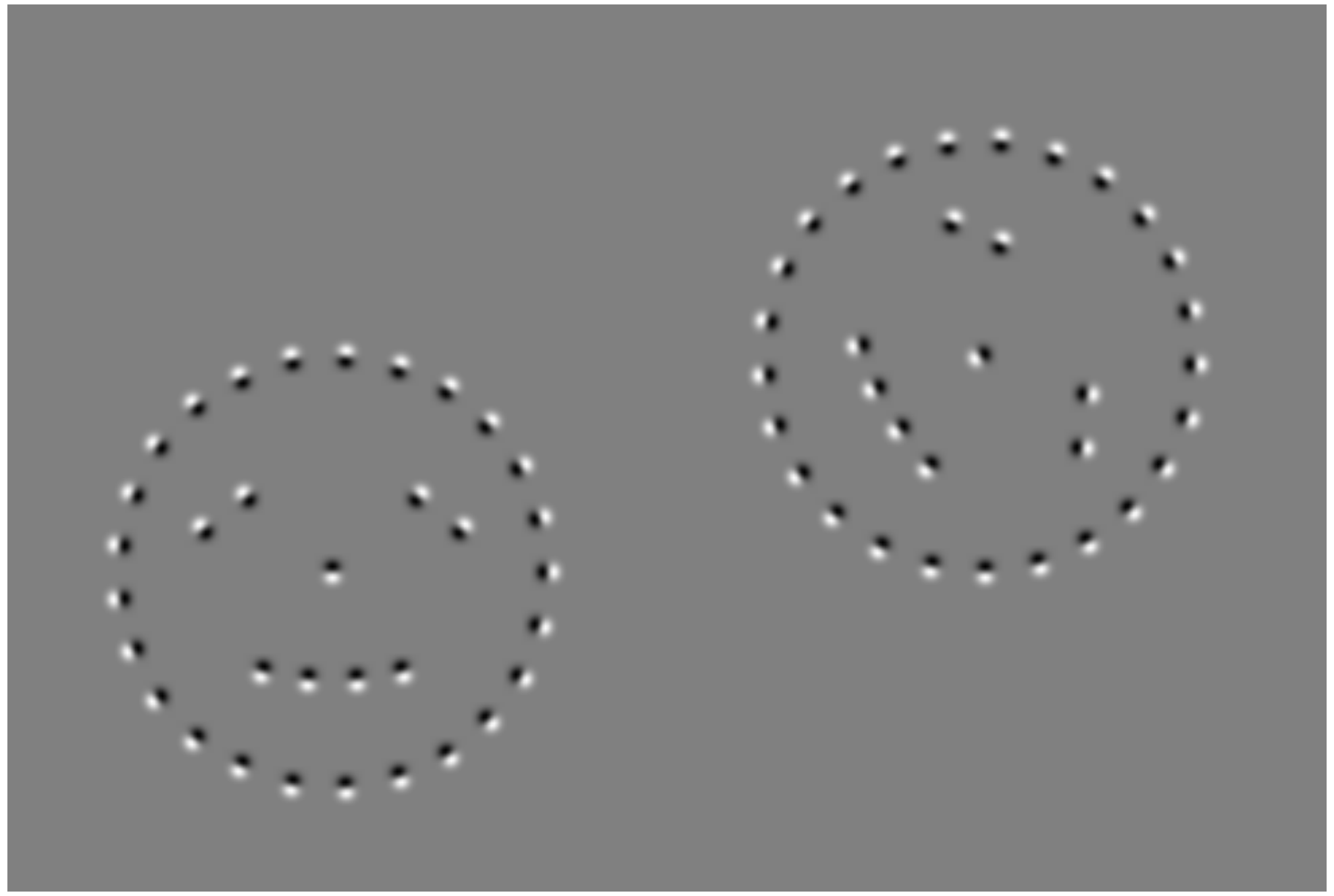
$$(k \tilde{\star} f)(g) = (\mathscr{L}_g^{G \to \mathbb{L}_2(X)} k \, , f)_{\mathbb{L}_2(X)} = (\mathscr{L}_{\mathbf{x}}^{(\mathbf{R}^d,+) \to \mathbb{L}_2(\mathbb{R}^d)} \mathscr{L}_h^{H \to \mathbb{L}_2(\mathbb{R}^d)} k \, , f)_{\mathbb{L}_2(\mathbb{R}^d)}$$

$$= \int_{\mathbb{R}^d} k(g^{-1}\mathbf{x}')f(\mathbf{x}')d\mathbf{x}' = \int_{\mathbb{R}^2} k_\theta(\mathbf{x}'-\mathbf{x})f(\mathbf{x}')d\mathbf{x}'$$

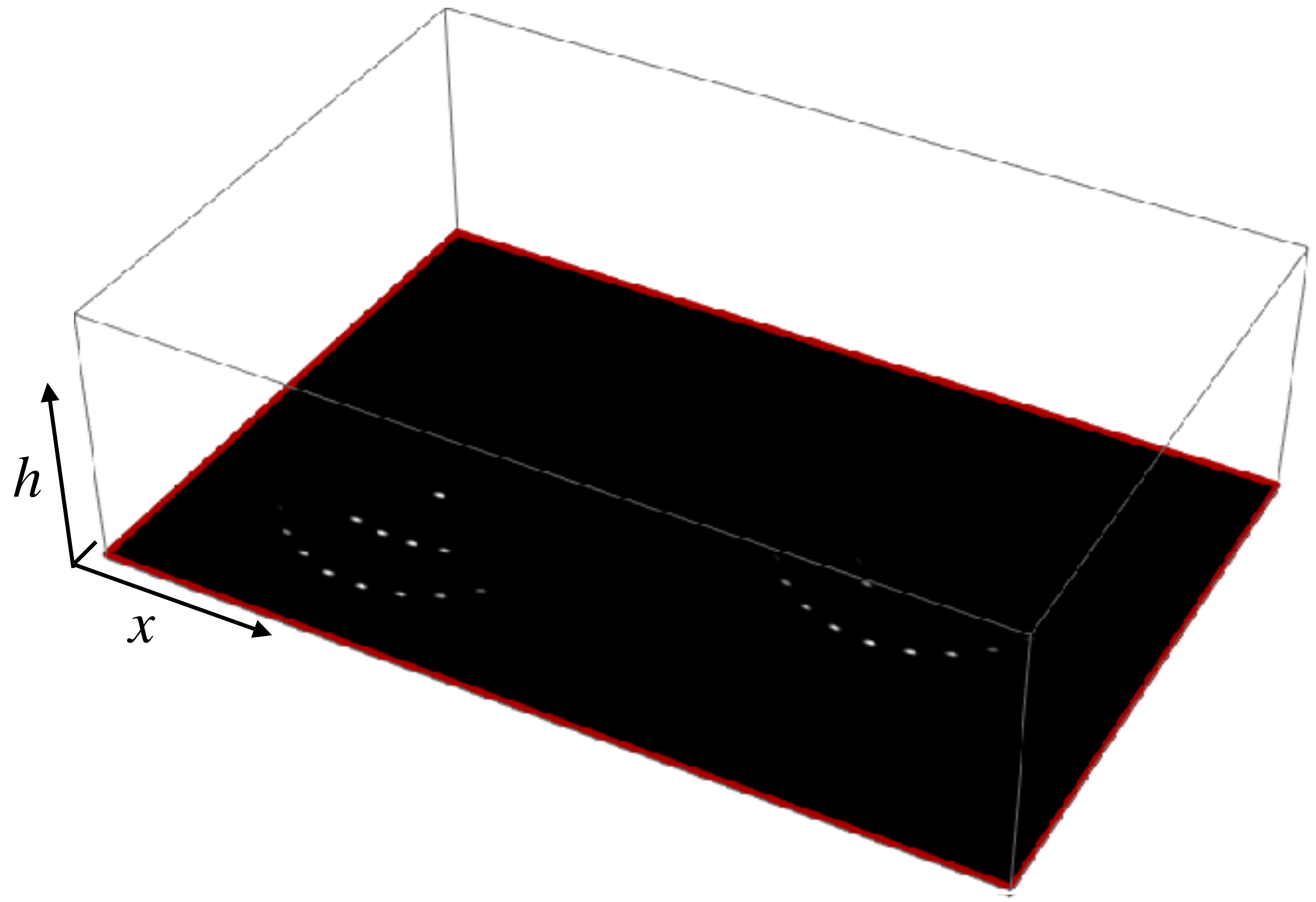
translation
"template matching"



$\star_{\mathbb{R}^2}$



=



Rotated
2D convolution kernel

2D input feature map

SE(2) output feature map (after ReLU)

Regular group convolutions revisited

Group convolution ($G = \mathbb{R}^d \rtimes H$):

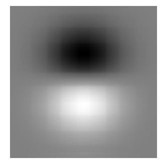
(e.g. $G = SE(2) = \mathbb{R}^2 \rtimes SO(2)$)
$X = \mathbb{R}^2$

$$(k \,\tilde{\star}\, f)(g) = (\mathcal{L}_g^{G\to\mathbb{L}_2(X)}\, k\,,\, f)_{\mathbb{L}_2(X)} = (\mathcal{L}_{\mathbf{x}}^{(\mathbf{R}^d,+)\to\mathbb{L}_2(\mathbb{R}^d)}\, \mathcal{L}_h^{H\to\mathbb{L}_2(\mathbb{R}^d)}\, k\,,\, f)_{\mathbb{L}_2(\mathbb{R}^d)}$$

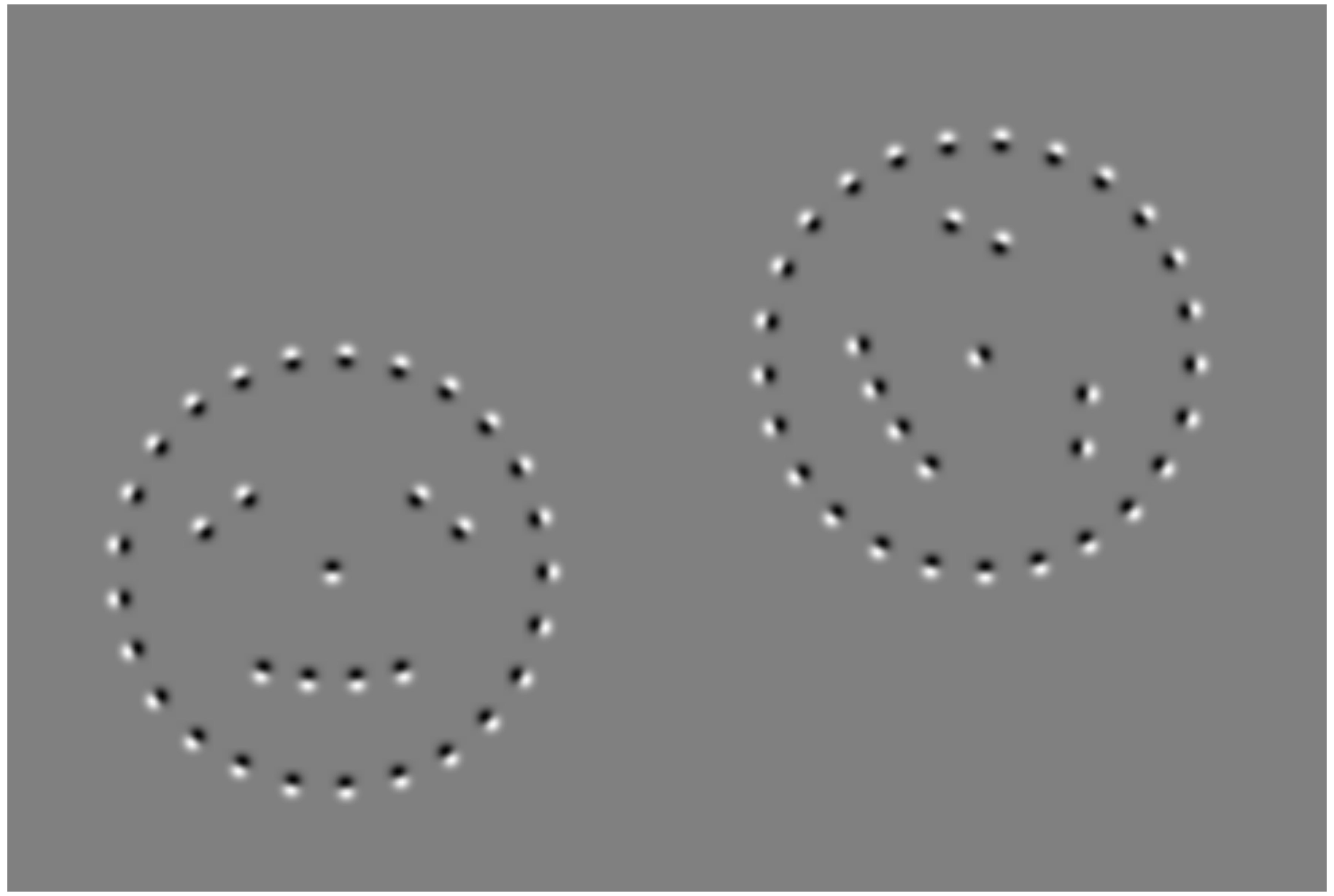
$$= \int_{\mathbb{R}^d} k(g^{-1}\mathbf{x}')f(\mathbf{x}')d\mathbf{x}' = \int_{\mathbb{R}^2} k_\theta(\mathbf{x}' - \mathbf{x})f(\mathbf{x}')d\mathbf{x}'$$

---

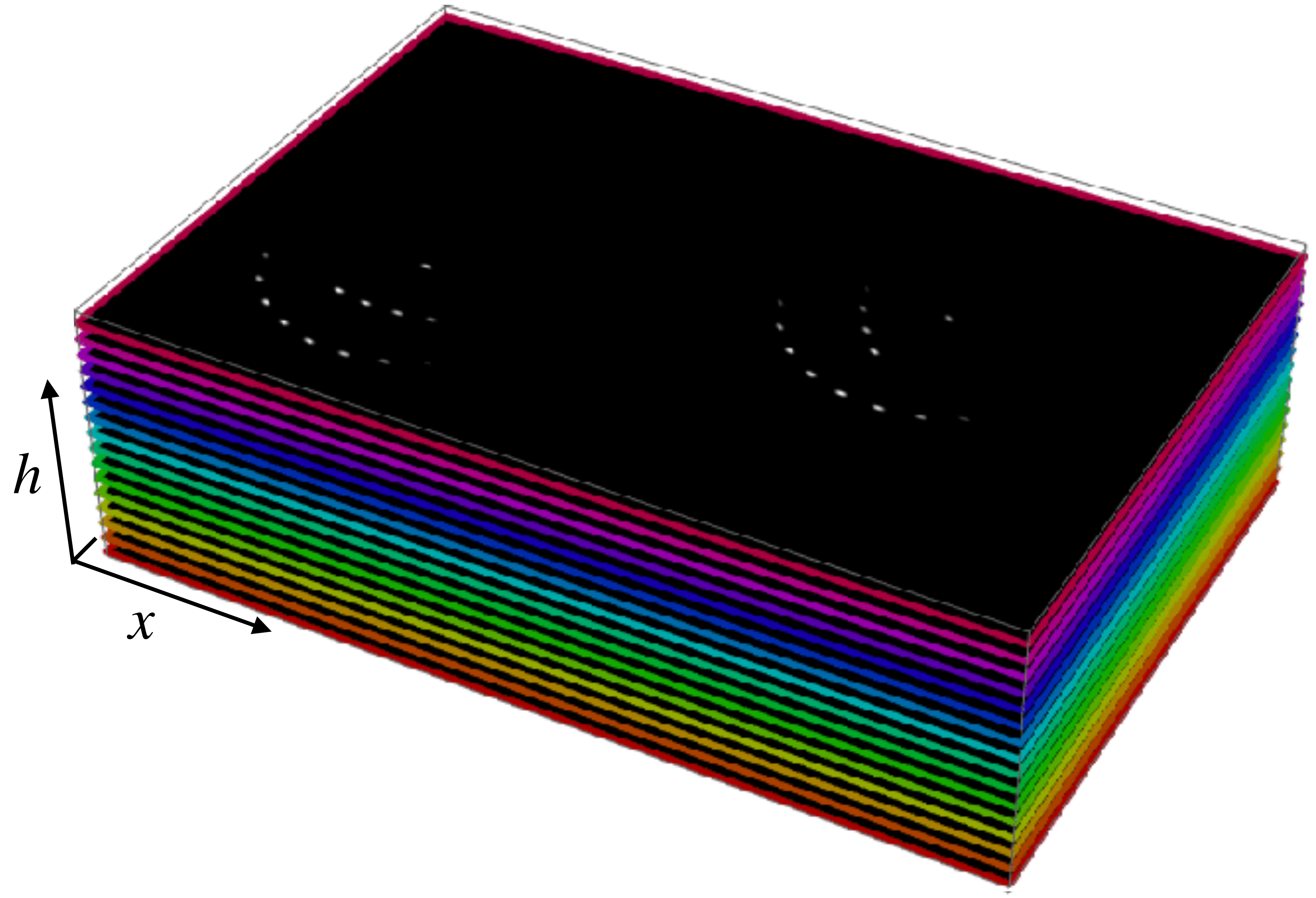
translation
“template matching”



$\star_{\mathbb{R}^2}$



=



Rotated
2D convolution kernel

2D input feature map

SE(2) output feature map (after ReLU)

Lifting convolution with steerable kernel

Group convolution ($G = \mathbb{R}^d \rtimes H$):

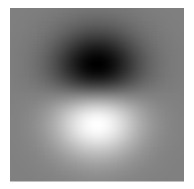
$$(k \,\tilde{\star}\, f)(\mathbf{x}, h) = \int_{\mathbb{R}^d} k(h^{-1}(\mathbf{x}' - \mathbf{x}) \,|\, \hat{\mathbf{w}}) f(\mathbf{x}') \mathrm{d}\mathbf{x}'$$

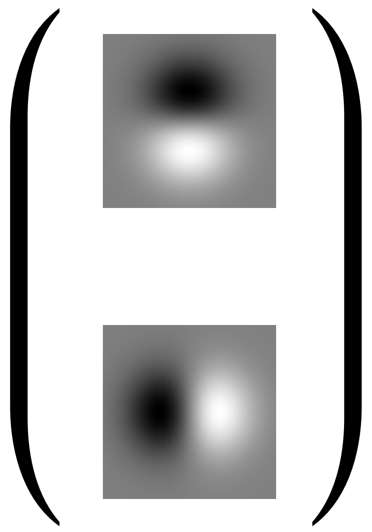
Lifting convolution with steerable kernel

Group convolution ($G = \mathbb{R}^d \rtimes H$):

$$(k \,\tilde{\star}\, f)(\mathbf{x}, h) = \int_{\mathbb{R}^d} k(h^{-1}(\mathbf{x}' - \mathbf{x}) \,|\, \hat{\mathbf{w}}) f(\mathbf{x}') \mathrm{d}\mathbf{x}'$$

$$k(\mathbf{x} \,|\, \hat{\mathbf{w}}) \quad = \quad \hat{\mathbf{w}}^{\dagger} \quad Y(\mathbf{x})$$



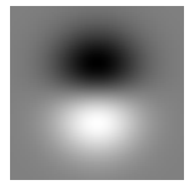


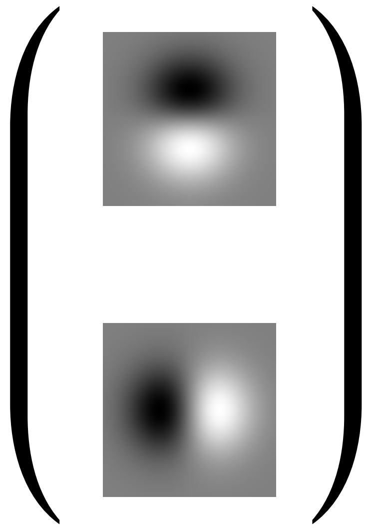
Lifting convolution with steerable kernel

Group convolution ($G = \mathbb{R}^d \rtimes H$):

$$(k \,\tilde{\star}\, f)(\mathbf{x}, h) = \int_{\mathbb{R}^d} k(h^{-1}(\mathbf{x}' - \mathbf{x}) \,|\, \hat{\mathbf{w}}) f(\mathbf{x}') \mathrm{d}\mathbf{x}'$$

$$k(h^{-1}\mathbf{x} \,|\, \hat{\mathbf{w}}) = (\rho(h)\hat{\mathbf{w}})^\dagger \; Y(\mathbf{x})$$





Lifting convolution with steerable kernel

Group convolution ($G = \mathbb{R}^d \rtimes H$):

$$(k \,\tilde{\star}\, f)(\mathbf{x}, h) = \int_{\mathbb{R}^d} k(h^{-1}(\mathbf{x}' - \mathbf{x}) \,|\, \hat{\mathbf{w}}) f(\mathbf{x}') d\mathbf{x}'$$

$$= \int_{\mathbb{R}^d} (\rho(h)\hat{\mathbf{w}})^{\dagger} Y(\mathbf{x}' - \mathbf{x}) f(\mathbf{x}') d\mathbf{x}'$$

$$k(h^{-1}\,\mathbf{x} \,|\, \hat{\mathbf{w}}) = (\rho(h)\hat{\mathbf{w}})^{\dagger} \quad Y(\mathbf{x})$$

Lifting convolution with steerable kernel

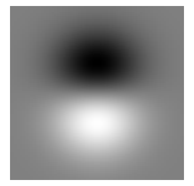
Group convolution ($G = \mathbb{R}^d \rtimes H$):

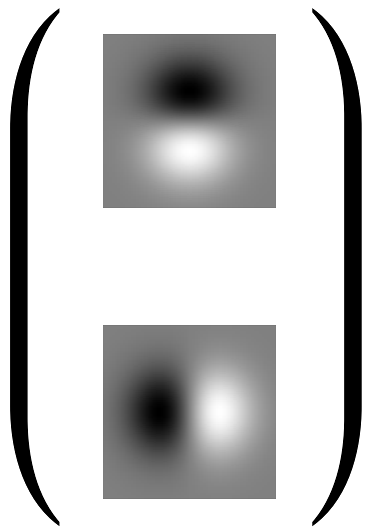
$$(k \,\tilde{\star}\, f)(\mathbf{x}, h) = \int_{\mathbb{R}^d} k(h^{-1}(\mathbf{x}' - \mathbf{x}) \,|\, \hat{\mathbf{w}}) f(\mathbf{x}') d\mathbf{x}'$$

$$= \int_{\mathbb{R}^d} (\rho(h)\hat{\mathbf{w}})^{\dagger} Y(\mathbf{x}' - \mathbf{x}) f(\mathbf{x}') d\mathbf{x}'$$

$$= (\rho(h)\, \hat{\mathbf{w}})^{\dagger} \int_{\mathbb{R}^d} Y(\mathbf{x}' - \mathbf{x}) f(\mathbf{x}') d\mathbf{x}'$$

$$k(h^{-1}\mathbf{x} \,|\, \hat{\mathbf{w}}) = (\rho(h)\hat{\mathbf{w}})^{\dagger} \quad Y(\mathbf{x})$$





Lifting convolution with steerable kernel

Group convolution ($G = \mathbb{R}^d \rtimes H$):

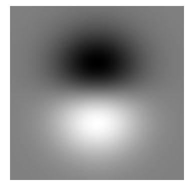
$$(k \,\tilde{\star}\, f)(\mathbf{x},h) = \int_{\mathbb{R}^d} k(h^{-1}(\mathbf{x}'-\mathbf{x}) \,|\, \hat{\mathbf{w}}) f(\mathbf{x}') d\mathbf{x}'$$

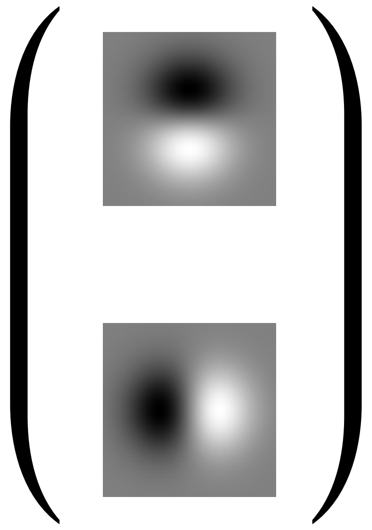
$$= \int_{\mathbb{R}^d} (\rho(h)\hat{\mathbf{w}})^\dagger Y(\mathbf{x}'-\mathbf{x}) f(\mathbf{x}') d\mathbf{x}'$$

$$= (\rho(h)\,\hat{\mathbf{w}})^\dagger \int_{\mathbb{R}^d} Y(\mathbf{x}'-\mathbf{x}) f(\mathbf{x}') d\mathbf{x}'$$

$$= (\rho(h)\,\hat{\mathbf{w}})^\dagger \, \hat{f}^Y(\mathbf{x})$$

$$k(h^{-1}\,\mathbf{x} \,|\, \hat{\mathbf{w}}) = (\rho(h)\hat{\mathbf{w}})^\dagger \quad Y(\mathbf{x})$$





Lifting convolution with steerable kernel

Group convolution ($G = \mathbb{R}^d \rtimes H$):

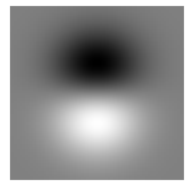
$$(k \,\tilde{\star}\, f)(\mathbf{x}, h) = \int_{\mathbb{R}^d} k(h^{-1}(\mathbf{x}' - \mathbf{x}) \,|\, \hat{\mathbf{w}}) f(\mathbf{x}') d\mathbf{x}'$$

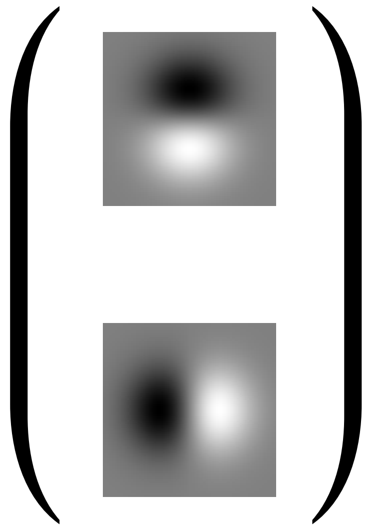
$$= \int_{\mathbb{R}^d} (\rho(h)\hat{\mathbf{w}})^\dagger Y(\mathbf{x}' - \mathbf{x}) f(\mathbf{x}') d\mathbf{x}'$$

$$= (\rho(h)\, \hat{\mathbf{w}})^\dagger \int_{\mathbb{R}^d} Y(\mathbf{x}' - \mathbf{x}) f(\mathbf{x}') d\mathbf{x}'$$

$$= (\rho(h)\, \hat{\mathbf{w}})^\dagger \hat{f}^Y(\mathbf{x})$$

$$k(h^{-1}\mathbf{x} \,|\, \hat{\mathbf{w}}) = (\rho(h)\hat{\mathbf{w}})^\dagger \quad Y(\mathbf{x})$$





! Freeman, W. T., & Adelson, E. H. (1991). The design and use of steerable filters. IEEE Transactions on Pattern analysis and machine intelligence, 13(9), 891-906.

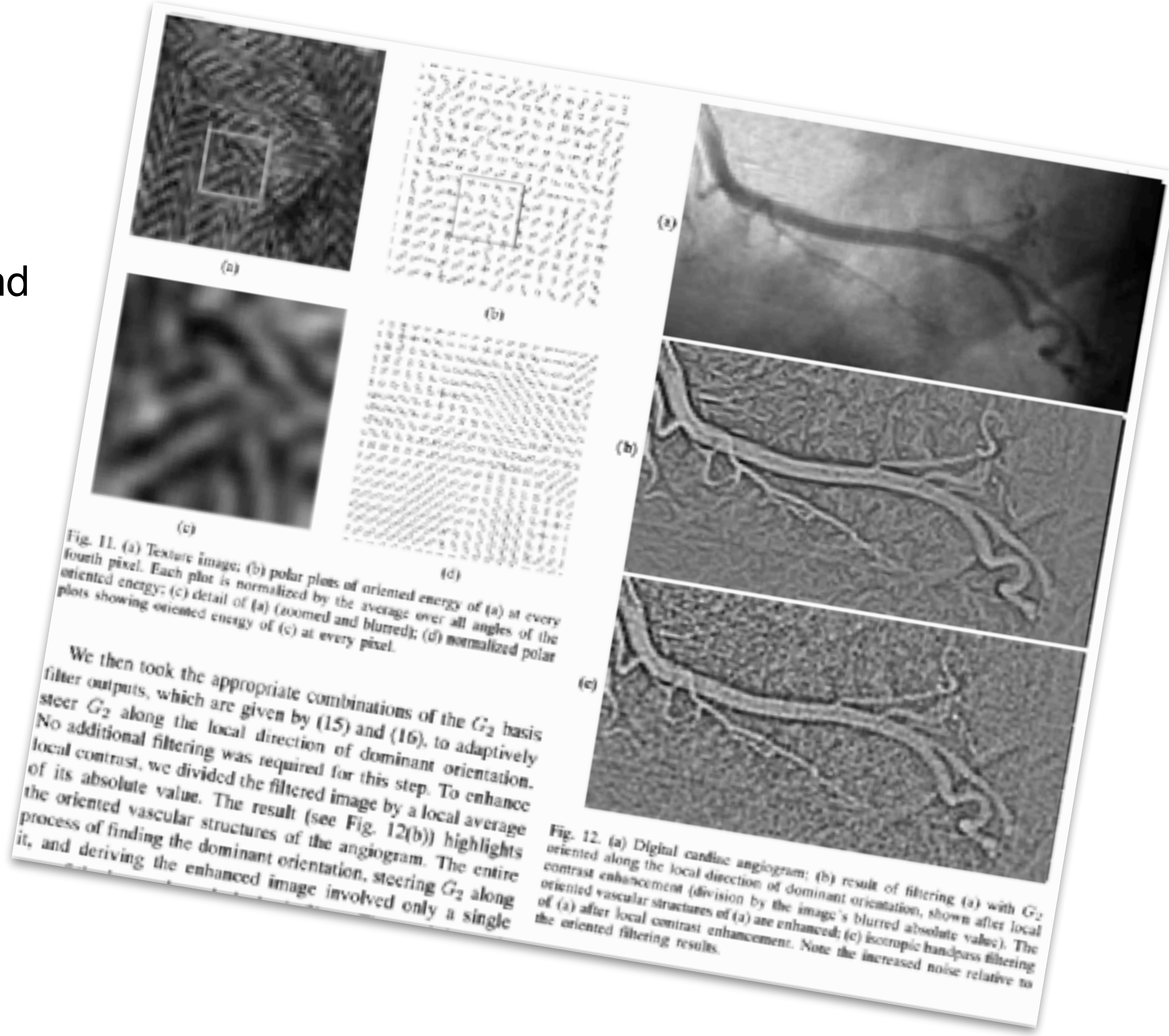


Fig. 11. (a) Texture image; (b) polar plots of oriented energy of (a) at every fourth pixel. Each plot is normalized by the average over all angles of the oriented energy; (c) detail of (a) (zoomed and blurred); (d) normalized polar plots showing oriented energy of (c) at every pixel.

We then took the appropriate combinations of the G_2 basis filter outputs, which are given by (15) and (16), to adaptively steer G_2 along the local direction of dominant orientation. No additional filtering was required for this step. To enhance local contrast, we divided the filtered image by a local average of its absolute value. The result (see Fig. 12(b)) highlights the oriented vascular structures of the angiogram. The entire process of finding the dominant orientation, steering G_2 along it, and deriving the enhanced image involved only a single

Fig. 12. (a) Digital cardiac angiogram; (b) result of filtering (a) with G_2 oriented along the local direction of dominant orientation, shown after local contrast enhancement (division by the image's blurred absolute value). The oriented vascular structures of (a) are enhanced; (c) isotropic bandpass filtering of (a) after local contrast enhancement. Note the increased noise relative to the oriented filtering results.

Lifting convolution with steerable kernel

Group convolution ($G = \mathbb{R}^d \rtimes H$):

(e.g. $G = SE(2) = \mathbb{R}^2 \rtimes SO(2)$)

$X = \mathbb{R}^2$

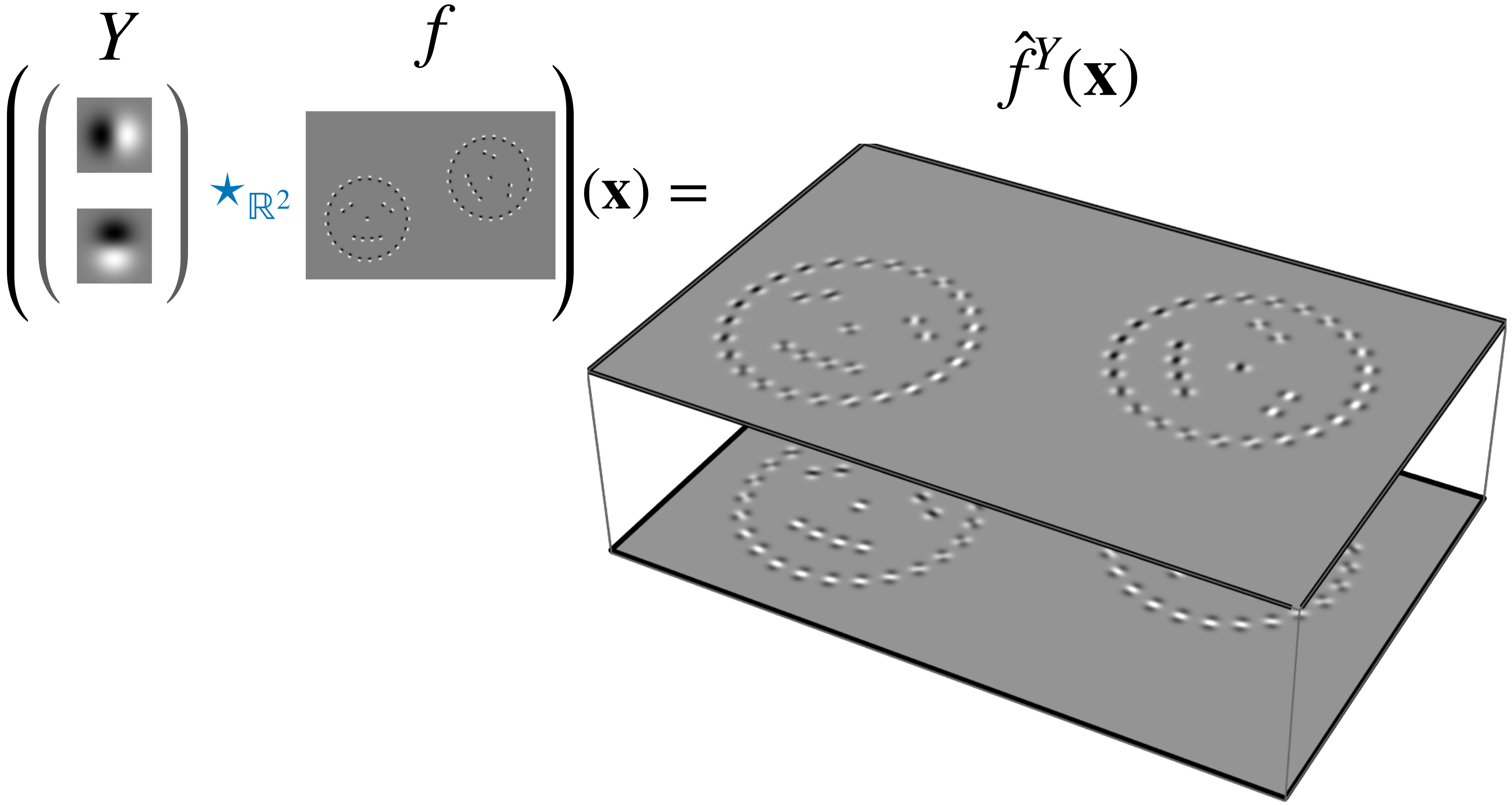
$$(k \,\tilde{\star}\, f)(\mathbf{x}, \theta) = (\rho(\mathbf{R}_\theta)\hat{\mathbf{w}})^\dagger \hat{f}^Y(\mathbf{x})$$

Lifting convolution with steerable kernel

Group convolution ($G = \mathbb{R}^d \rtimes H$):

(e.g. $G = SE(2) = \mathbb{R}^2 \rtimes SO(2)$)
$X = \mathbb{R}^2$

$$(k \,\tilde{\star}\, f)(\mathbf{x}, \theta) = (\rho(\mathbf{R}_\theta)\hat{\mathbf{w}})^\dagger \hat{f}^Y(\mathbf{x})$$

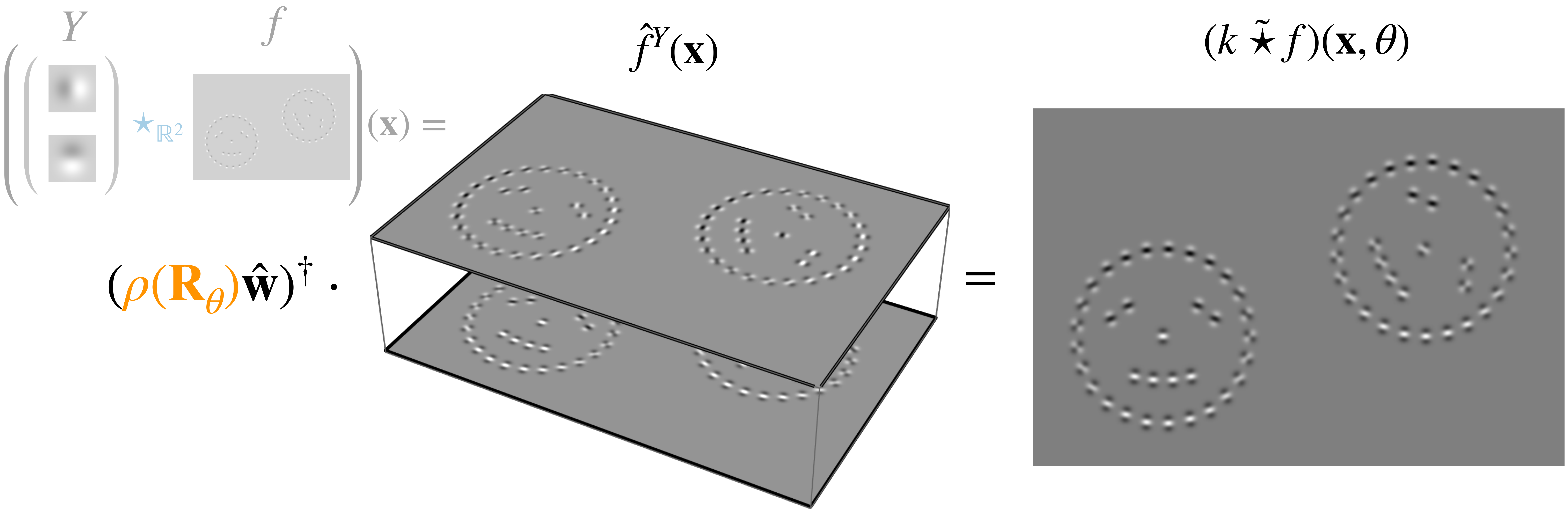


Lifting convolution with steerable kernel

Group convolution ($G = \mathbb{R}^d \rtimes H$):

$$(k \,\tilde{\star}\, f)(\mathbf{x}, \theta) \;=\; (\rho(\mathbf{R}_\theta)\hat{\mathbf{w}})^\dagger \, \hat{f}^Y(\mathbf{x})$$

(e.g. $G = SE(2) = \mathbb{R}^2 \rtimes SO(2)$)
$X = \mathbb{R}^2$



Lifting convolution with steerable kernel

Group convolution ($G = \mathbb{R}^d \rtimes H$):

$$(k \,\tilde{\star}\, f)(\mathbf{x},h) = \int_{\mathbb{R}^d} k(h^{-1}(\mathbf{x}'-\mathbf{x}) \,|\, \hat{\mathbf{w}}) f(\mathbf{x}')\mathrm{d}\mathbf{x}'$$

$$= \int_{\mathbb{R}^d} (\rho(h)\hat{\mathbf{w}})^\dagger Y(\mathbf{x}'-\mathbf{x}) f(\mathbf{x}')\mathrm{d}\mathbf{x}'$$

$$= (\rho(h)\,\hat{\mathbf{w}})^\dagger \int_{\mathbb{R}^d} Y(\mathbf{x}'-\mathbf{x}) f(\mathbf{x}')\mathrm{d}\mathbf{x}'$$

$$= (\rho(h)\,\hat{\mathbf{w}})^\dagger \, \hat{f}^Y(\mathbf{x})$$

$$k(h^{-1}\,\mathbf{x} \,|\, \hat{\mathbf{w}}) = (\rho(h^{-1})\hat{\mathbf{w}})^T \; Y(\mathbf{x})$$

! Freeman, W. T., & Adelson, E. H. (1991). The design and use of steerable filters. IEEE Transactions on Pattern analysis and machine intelligence, 13(9), 891-906.

Lifting convolution with steerable kernel

Group convolution ($G = \mathbb{R}^d \rtimes H$):

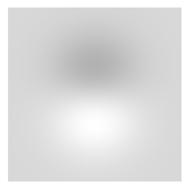
$$(k \,\tilde{\star}\, f)(\mathbf{x}, h) = \int_{\mathbb{R}^d} k(h^{-1}(\mathbf{x}' - \mathbf{x}) \,|\, \hat{\mathbf{w}}) f(\mathbf{x}') d\mathbf{x}'$$

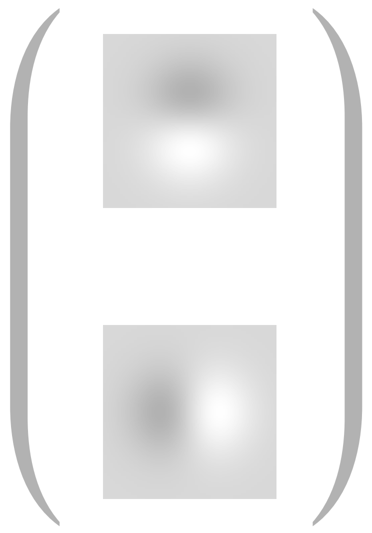
$$= \int_{\mathbb{R}^d} (\rho(h)\hat{\mathbf{w}})^{\dagger} Y(\mathbf{x}' - \mathbf{x}) f(\mathbf{x}') d\mathbf{x}'$$

$$= (\rho(h)\, \hat{\mathbf{w}})^{\dagger} \int_{\mathbb{R}^d} Y(\mathbf{x}' - \mathbf{x}) f(\mathbf{x}') d\mathbf{x}'$$

$$= (\rho(h)\, \hat{\mathbf{w}})^{\dagger}\, \hat{f}^{Y}(\mathbf{x})$$

$$k(h^{-1}\,\mathbf{x} \,|\, \hat{\mathbf{w}}) = (\rho(h^{-1})\hat{\mathbf{w}})^{T}\; Y(\mathbf{x})$$





Freeman, W. T., & Adelson, E. H. (1991). The design and use of steerable filters. IEEE Transactions on Pattern analysis and machine intelligence, 13(9), 891-906.

Lifting convolution with steerable kernel

Group convolution ($G = \mathbb{R}^d \rtimes H$):

$$(k \,\tilde{\star}\, f)(\mathbf{x},h) = \int_{\mathbb{R}^d} k(h^{-1}(\mathbf{x}'-\mathbf{x}) \,|\, \hat{\mathbf{w}}) f(\mathbf{x}') d\mathbf{x}'$$

$$= \int_{\mathbb{R}^d} (\rho(h)\hat{\mathbf{w}})^\dagger Y(\mathbf{x}'-\mathbf{x}) f(\mathbf{x}') d\mathbf{x}'$$

$$= (\rho(h)\, \hat{\mathbf{w}})^\dagger \int_{\mathbb{R}^d} Y(\mathbf{x}'-\mathbf{x}) f(\mathbf{x}') d\mathbf{x}'$$

$$= (\rho(h)\, \hat{\mathbf{w}})^\dagger \, \hat{f}^Y(\mathbf{x})$$

$$= \mathrm{tr}(\, \hat{f}^Y(\mathbf{x})\, \hat{\mathbf{w}}^\dagger \, \rho(h^{-1}) \,)$$

$$k(h^{-1}\, \mathbf{x} \,|\, \hat{\mathbf{w}}) = (\rho(h^{-1})\hat{\mathbf{w}})^T \; Y(\mathbf{x})$$

! Freeman, W. T., & Adelson, E. H. (1991). The design and use of steerable filters. IEEE Transactions on Pattern analysis and machine intelligence, 13(9), 891-906.

$\mathbf{a}^T \mathbf{b} = \mathrm{tr}(\mathbf{b}\, \mathbf{a}^T)$ and $\rho(h)^\dagger = \rho(h^{-1})$

Lifting convolution with steerable kernel

Group convolution ($G = \mathbb{R}^d \rtimes H$):

$$(k \,\tilde{\star}\, f)(\mathbf{x}, h) = \int_{\mathbb{R}^d} k(h^{-1}(\mathbf{x}' - \mathbf{x}) \,|\, \hat{\mathbf{w}}) f(\mathbf{x}') \mathrm{d}\mathbf{x}'$$

$$= \int_{\mathbb{R}^d} (\rho(h)\hat{\mathbf{w}})^{\dagger} Y(\mathbf{x}' - \mathbf{x}) f(\mathbf{x}') \mathrm{d}\mathbf{x}'$$

$$= (\rho(h)\, \hat{\mathbf{w}})^{\dagger} \int_{\mathbb{R}^d} Y(\mathbf{x}' - \mathbf{x}) f(\mathbf{x}') \mathrm{d}\mathbf{x}'$$

$$= (\rho(h)\, \hat{\mathbf{w}})^{\dagger}\, \hat{f}^{Y}(\mathbf{x})$$

$$= \mathrm{tr}(\, \hat{f}^{Y}(\mathbf{x})\, \hat{\mathbf{w}}^{\dagger}\, \rho(h^{-1})\,)$$

$$= \mathrm{tr}(\, \hat{f}(\mathbf{x})\, \rho(h^{-1})\,)$$

$$k(h^{-1}\mathbf{x} \,|\, \hat{\mathbf{w}}) = (\rho(h^{-1})\hat{\mathbf{w}})^{T}\, Y(\mathbf{x})$$

! Freeman, W. T., & Adelson, E. H. (1991). The design and use of steerable filters. IEEE Transactions on Pattern analysis and machine intelligence, 13(9), 891-906.

$\mathbf{a}^{T}\mathbf{b} = \mathrm{tr}(\mathbf{b}\,\mathbf{a}^{T})$ and $\rho(h)^{\dagger} = \rho(h^{-1})$

$$\hat{f}(\mathbf{x}) = \hat{f}^{Y}(\mathbf{x})\, \hat{w}^{\dagger}$$

Lifting convolution with steerable kernel

Group convolution ($G = \mathbb{R}^d \rtimes H$):

$$(k \tilde{\star} f)(\mathbf{x}, h) = \int_{\mathbb{R}^d} k(h^{-1}(\mathbf{x}' - \mathbf{x}) \,|\, \hat{\mathbf{w}}) f(\mathbf{x}') d\mathbf{x}'$$

$$= \int_{\mathbb{R}^d} (\rho(h)\hat{\mathbf{w}})^\dagger Y(\mathbf{x}' - \mathbf{x}) f(\mathbf{x}') d\mathbf{x}'$$

$$= (\rho(h)\, \hat{\mathbf{w}})^\dagger \int_{\mathbb{R}^d} Y(\mathbf{x}' - \mathbf{x}) f(\mathbf{x}') d\mathbf{x}'$$

$$= (\rho(h)\, \hat{\mathbf{w}})^\dagger\, \hat{f}^Y(\mathbf{x})$$

$$= \mathrm{tr}(\, \hat{f}^Y(\mathbf{x})\, \hat{\mathbf{w}}^\dagger\, \rho(h^{-1})\,)$$

$$= \mathrm{tr}(\, \hat{f}(\mathbf{x})\, \rho(h^{-1})\,)$$

$$= \mathscr{F}_H^{-1}[\, \hat{f}(\mathbf{x})\,](h)$$

$$k(h^{-1}\mathbf{x} \,|\, \hat{\mathbf{w}}) = (\rho(h^{-1})\hat{\mathbf{w}})^T\; Y(\mathbf{x})$$

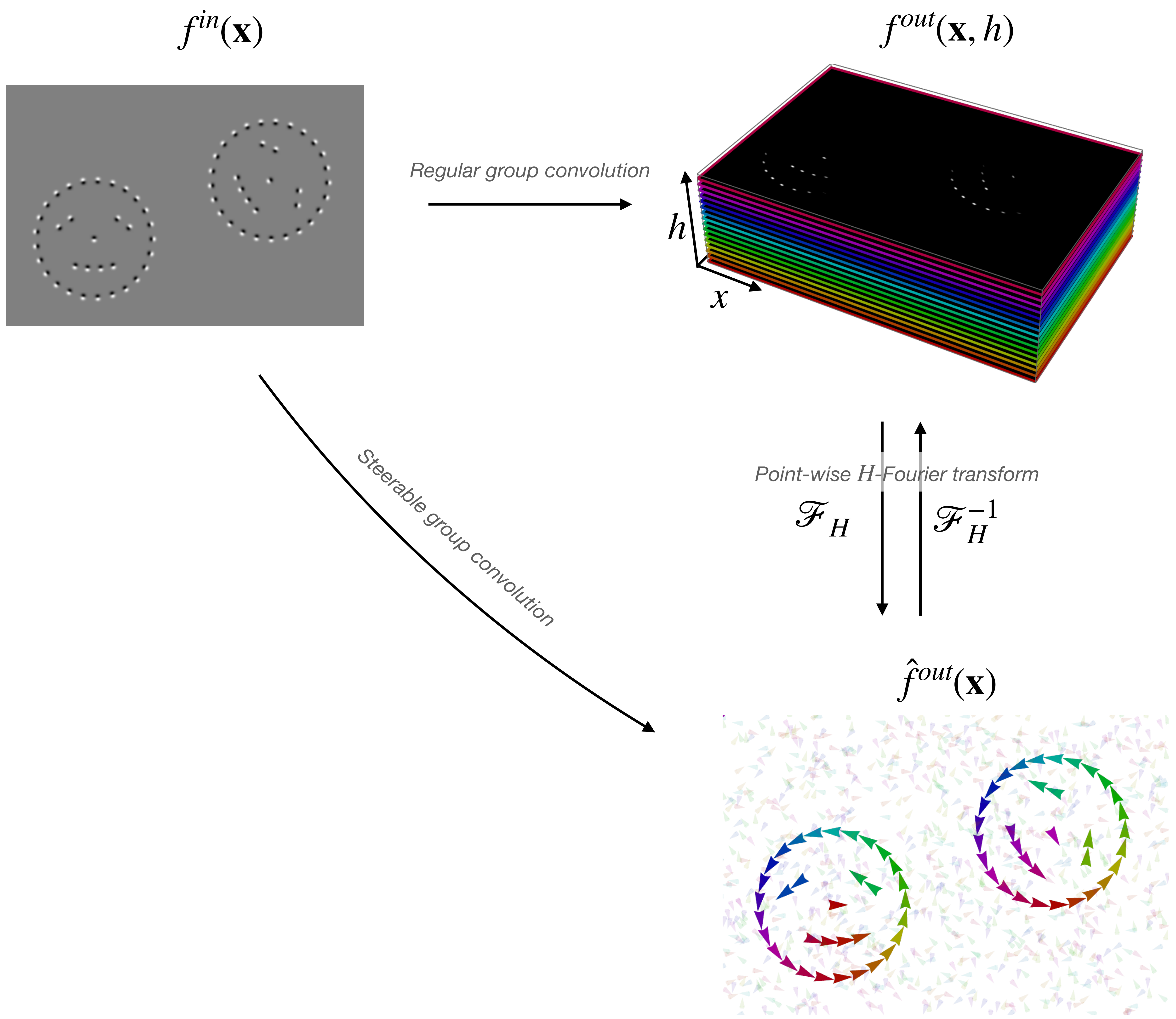
! Freeman, W. T., & Adelson, E. H. (1991). The design and use of steerable filters. IEEE Transactions on Pattern analysis and machine intelligence, 13(9), 891-906.

$\mathbf{a}^T \mathbf{b} = \mathrm{tr}(\mathbf{b}\, \mathbf{a}^T)$ and $\rho(h)^\dagger = \rho(h^{-1})$

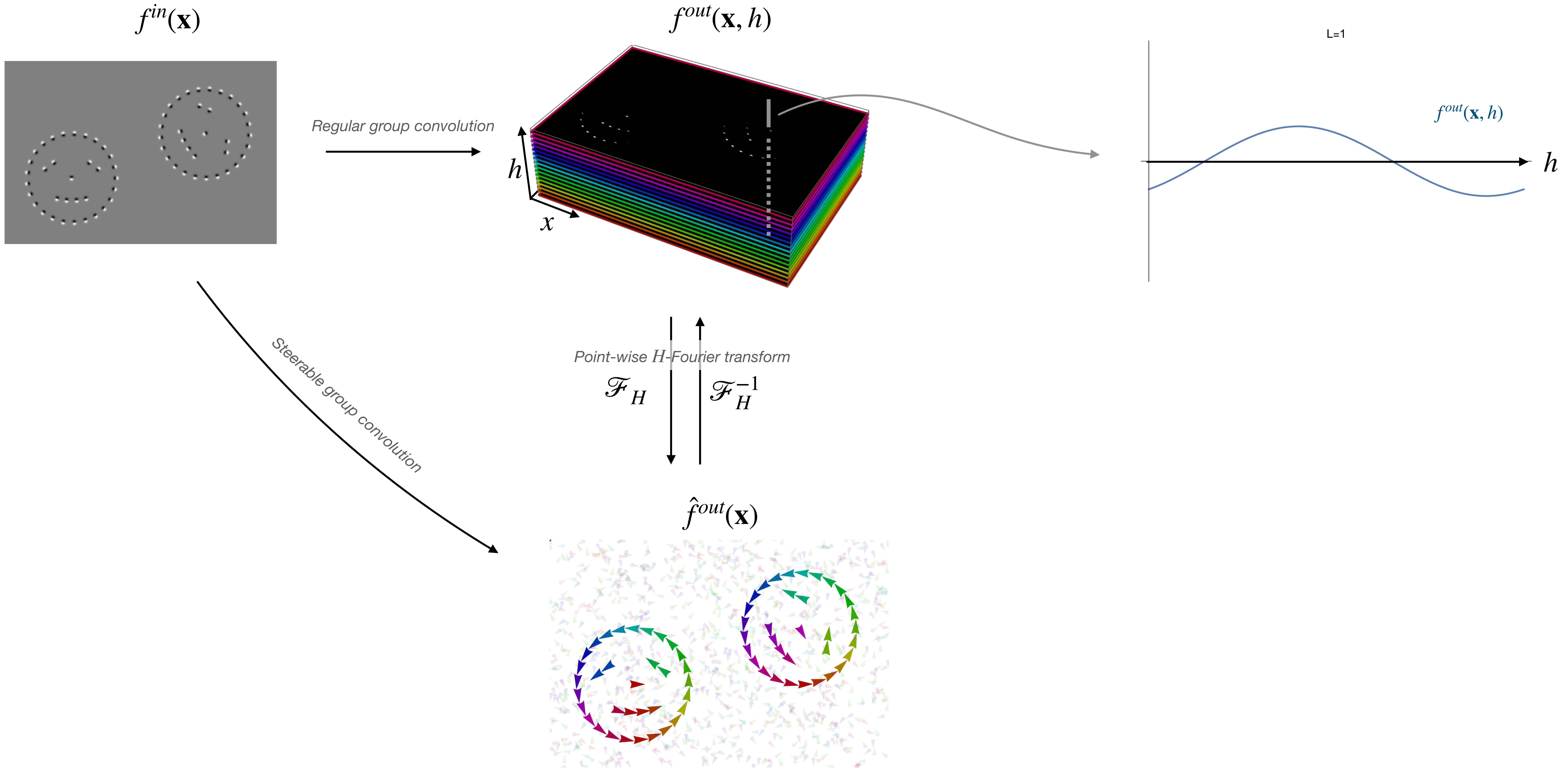
$\hat{f}(\mathbf{x}) = \hat{f}^Y(\mathbf{x})\, \hat{w}^\dagger$

Inverse H-Fourier transform!

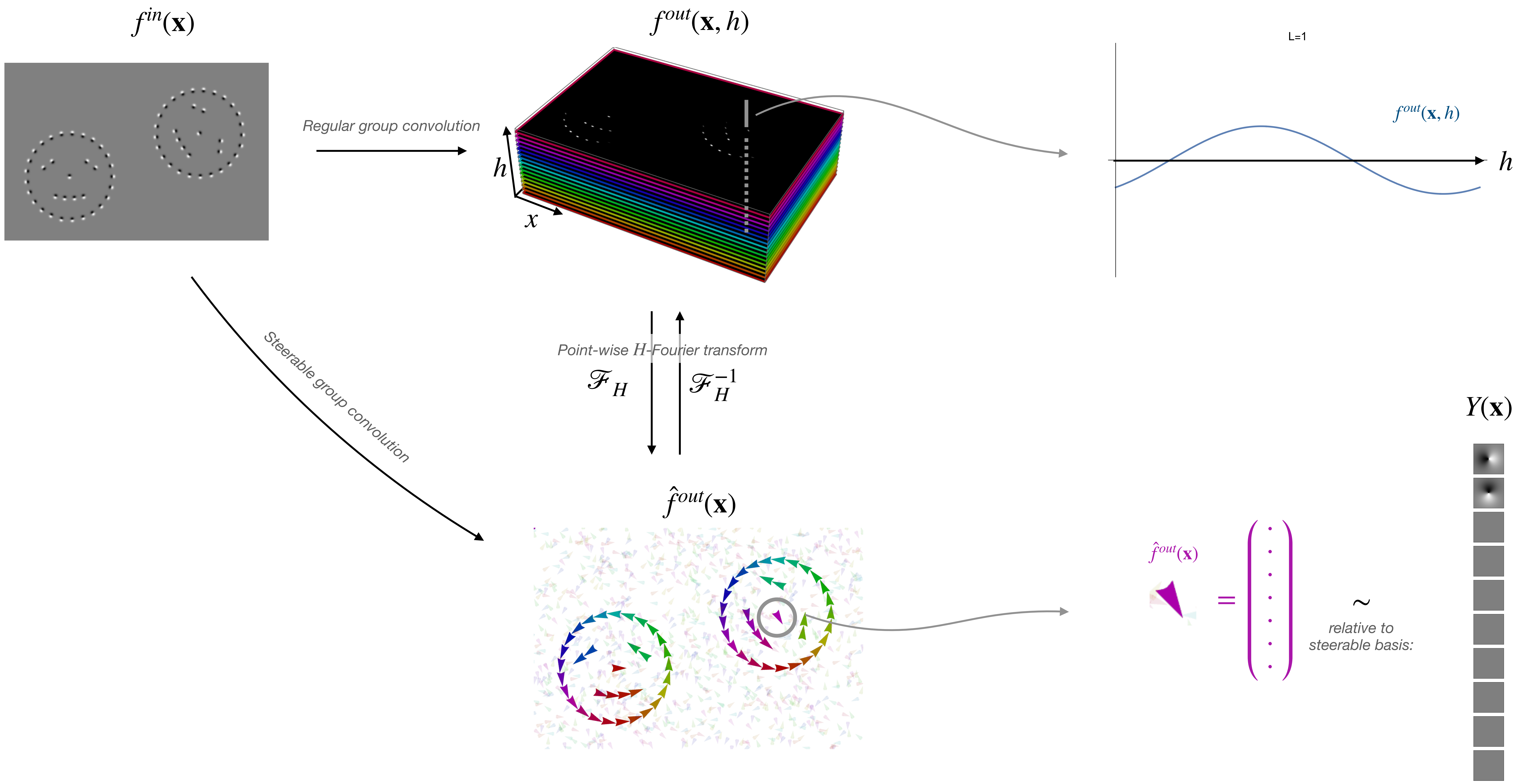
From regular to steerable via a Fourier transform



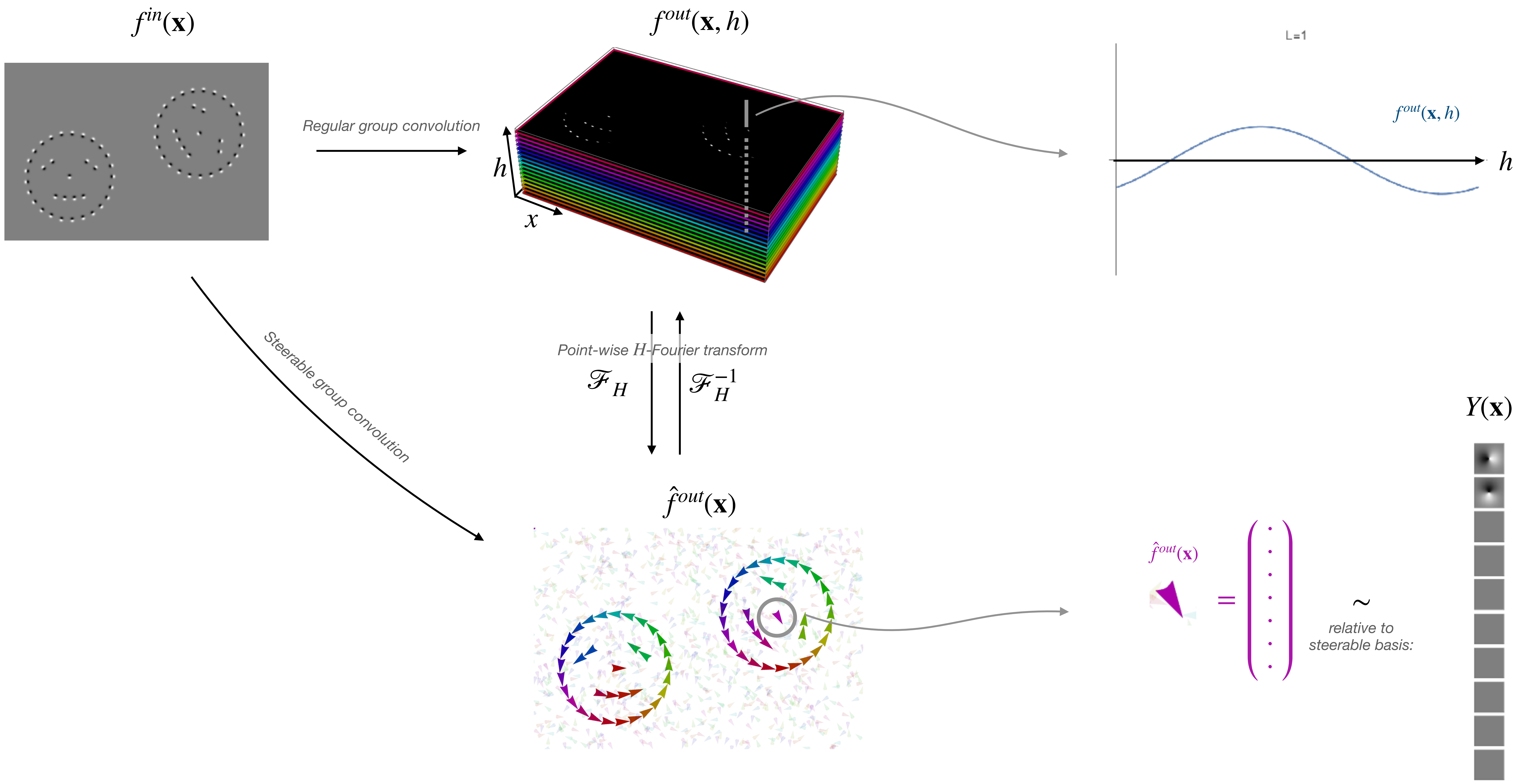
From regular to steerable via a Fourier transform



From regular to steerable via a Fourier transform



From regular to steerable via a Fourier transform



From regular to steerable via a Fourier transform

Regular group convolutions:
Domain expanded feature maps

$$f^{(l)} : \mathbb{R}^d \times H \rightarrow \mathbb{R}$$

added axis

Steerable group convolutions:
Co-domain expanded feature maps (feature fields)

$$\hat{f}^{(l)} : \mathbb{R}^d \rightarrow V_H$$

vector field instead of scalar field
(vectors in V_H transform via group H representations)