UNIVERSITY OF AMSTERDAM

AMLAB Amsterdam Machine Learning Lab

Group Equivariant Deep Learning

Lecture 1 - Regular group convolutions

Lecture 1.6 - Group Theory | Homogeneous/quotient spaces

Preliminaries for "group convolutions are all you need"

Transitive action, homogeneous space, quotient space, examples

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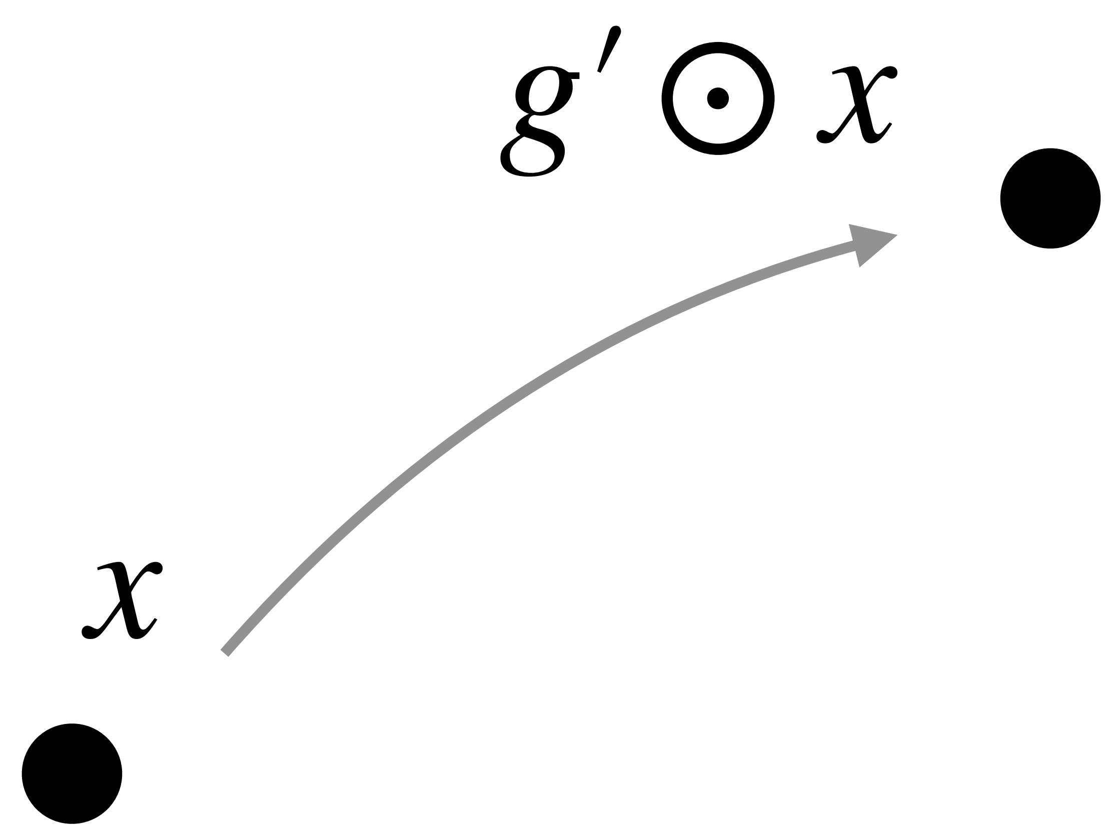
This mini-course serves as a module with the UvA Master AI course Deep Learning 2 https://uvadl2c.github.io/

Group action

Group action: An operator $\odot : G \times X \to X$ such that

$$\forall_{g,g' \in G, x \in X} : \; g \odot (g' \odot x) = (gg') \odot x$$



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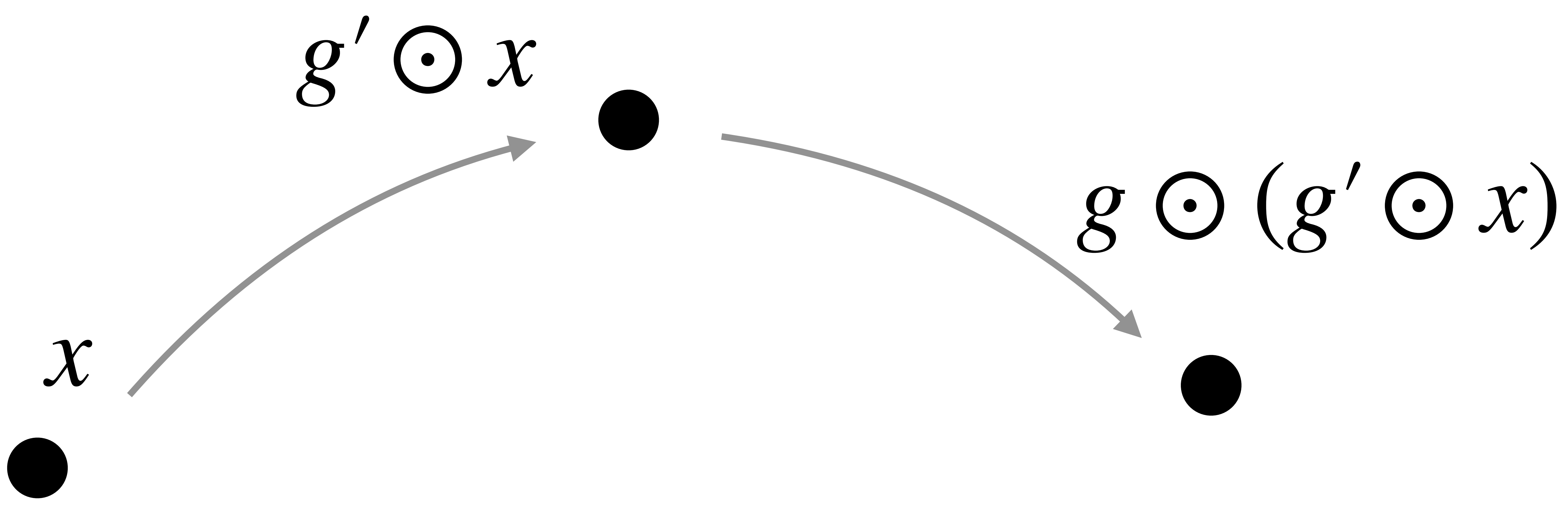

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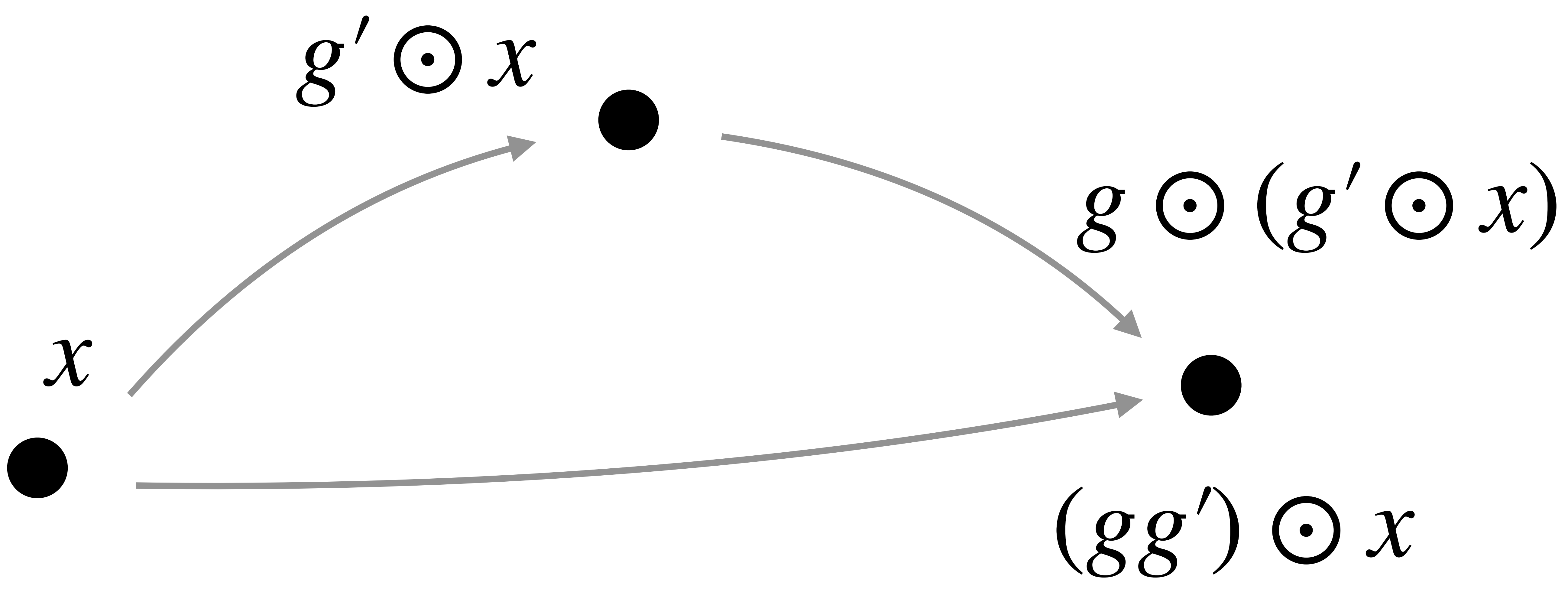

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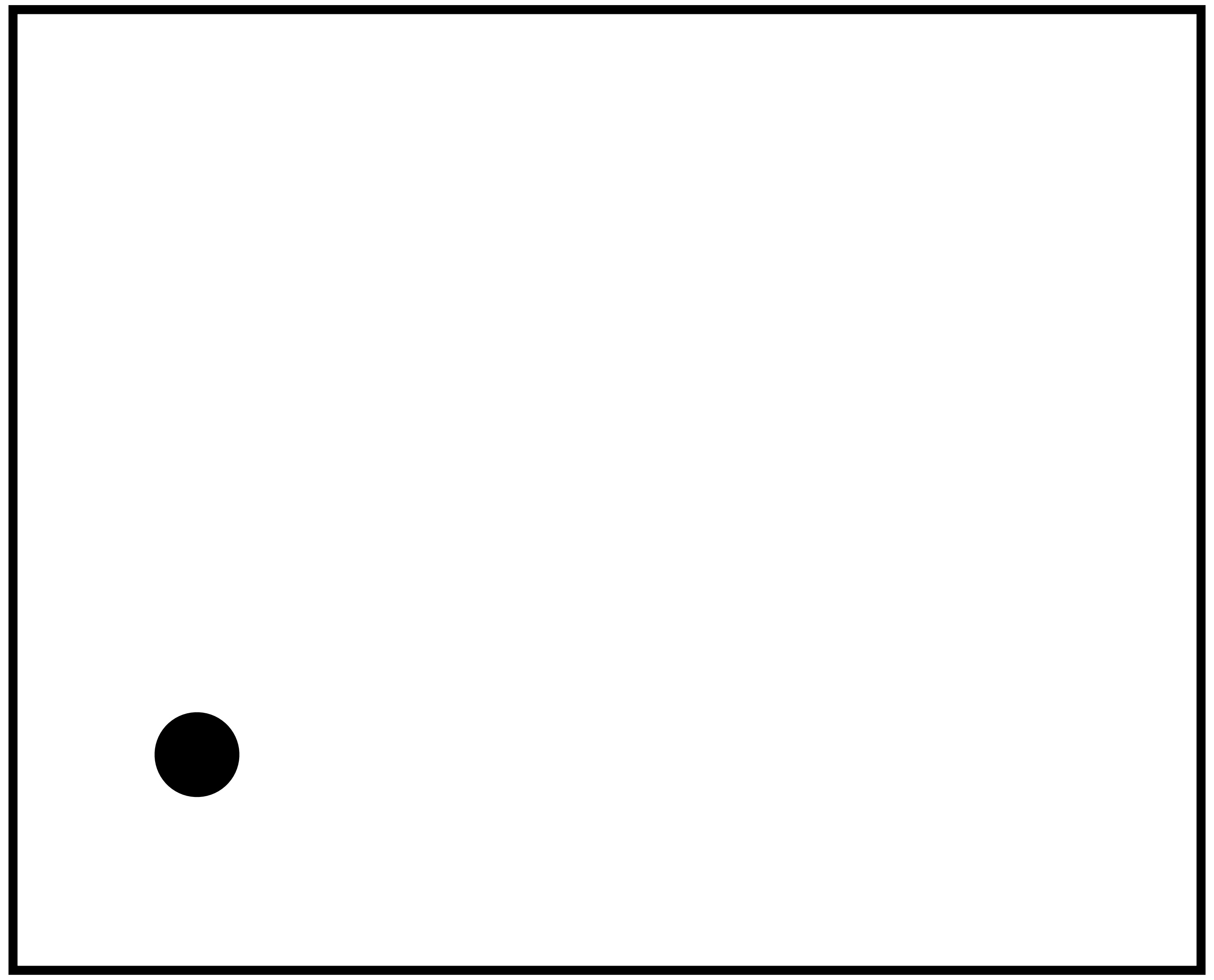

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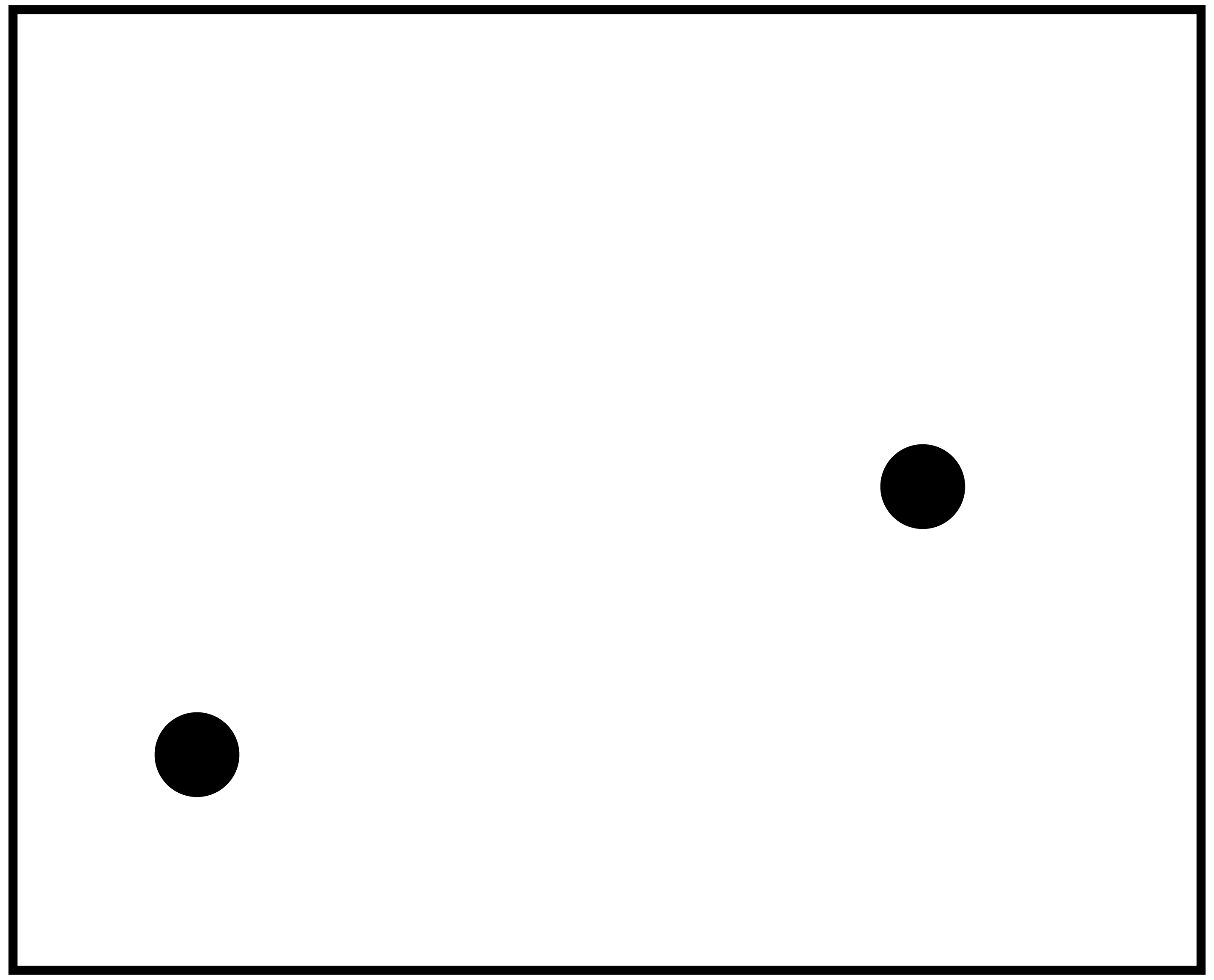

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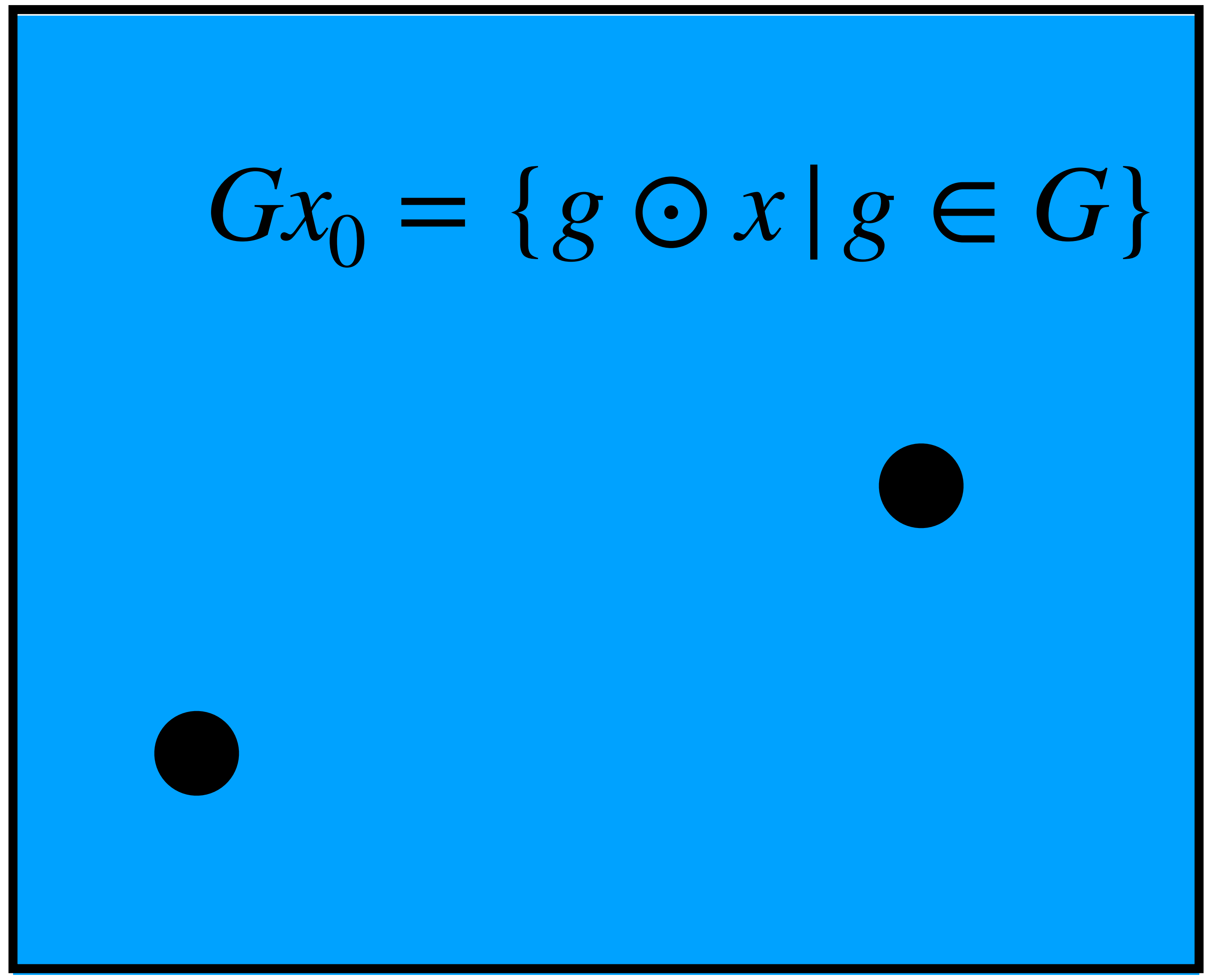

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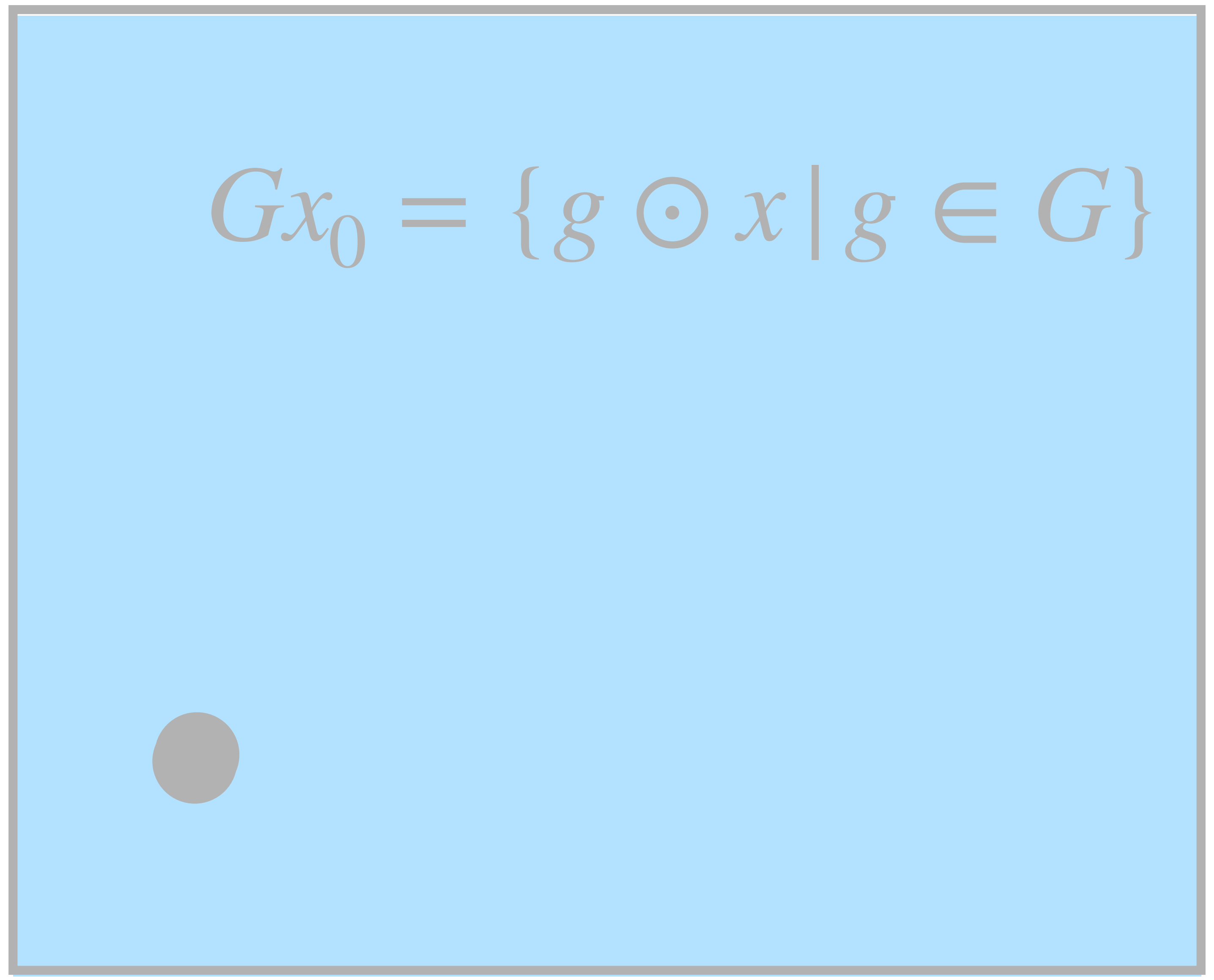

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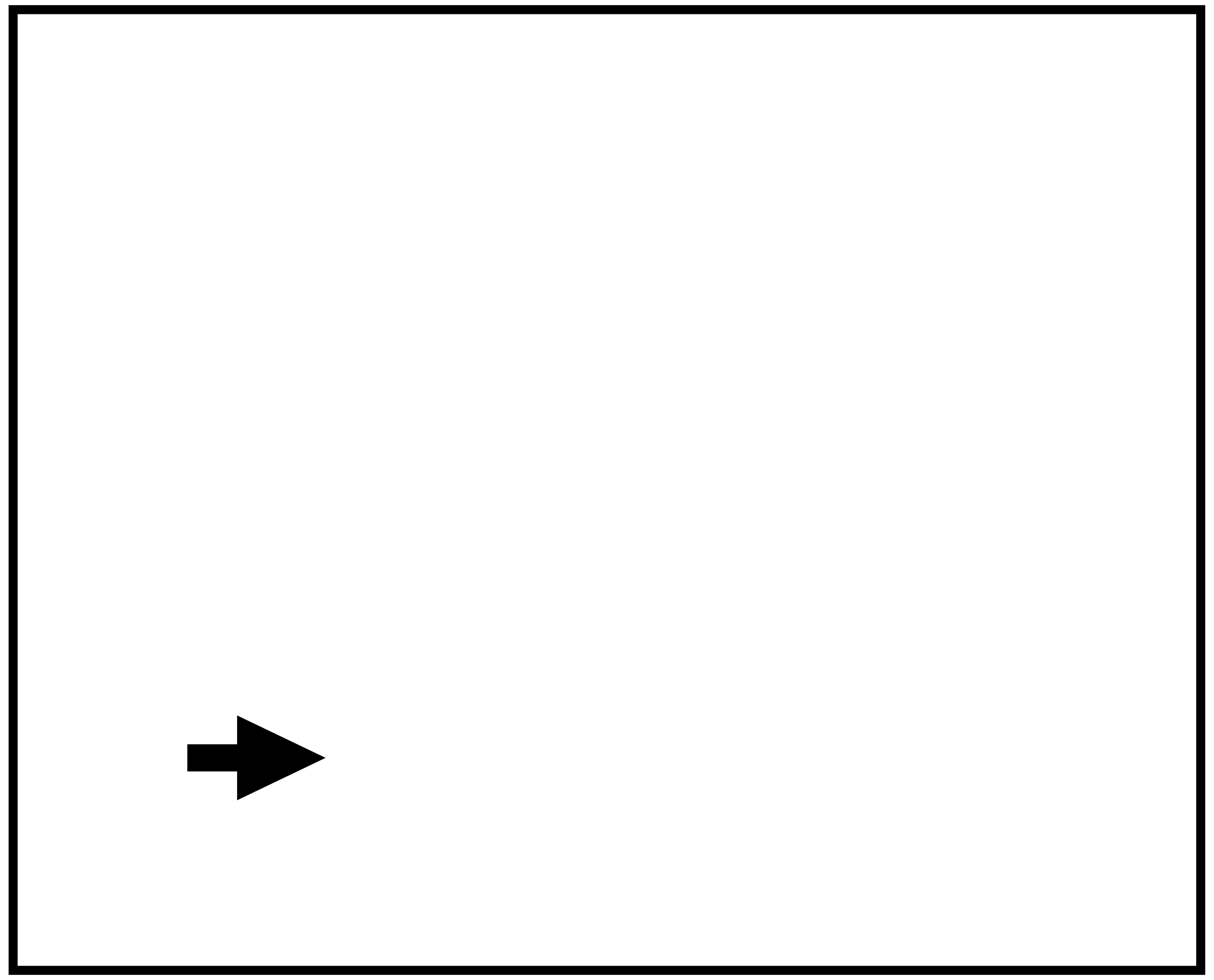

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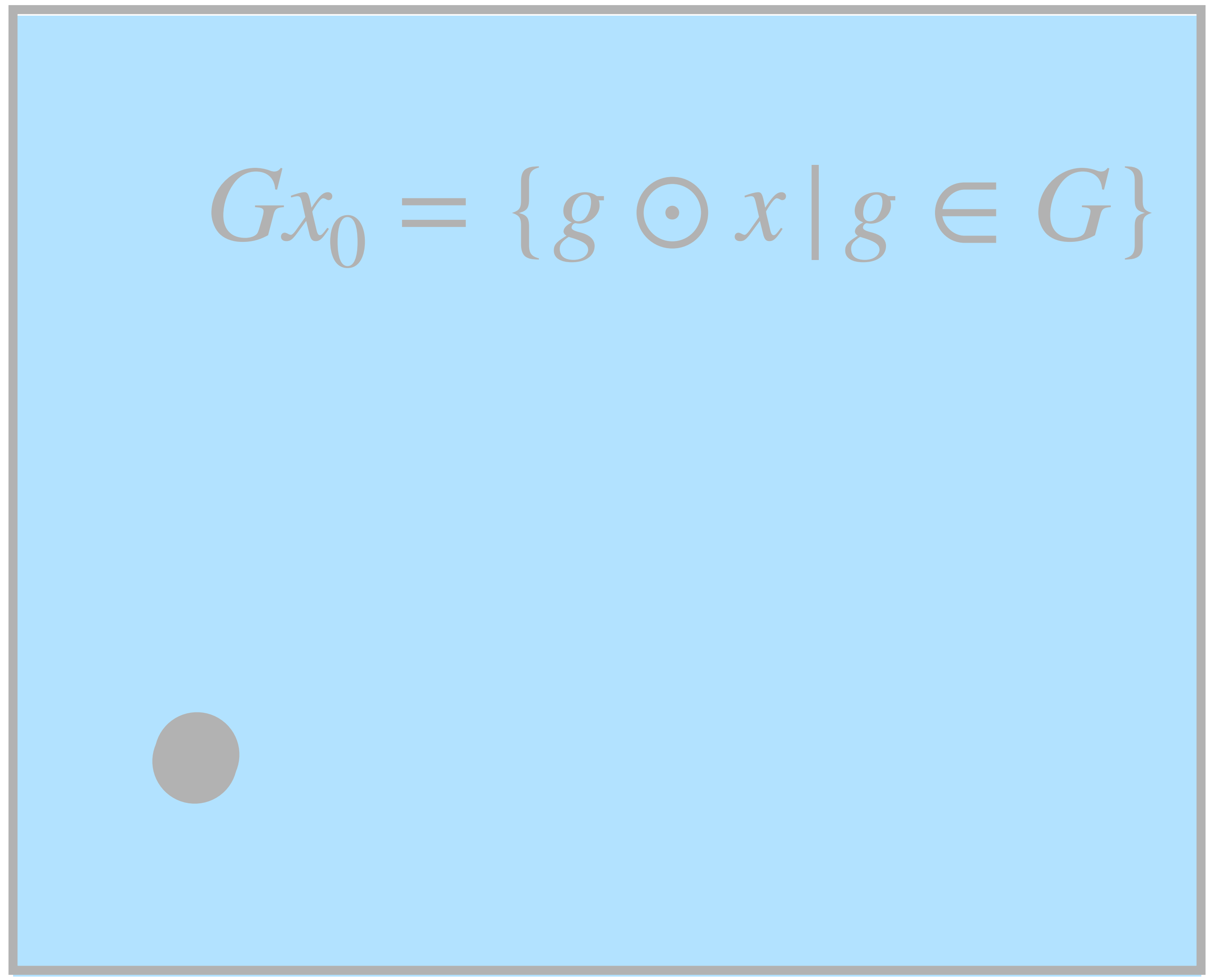

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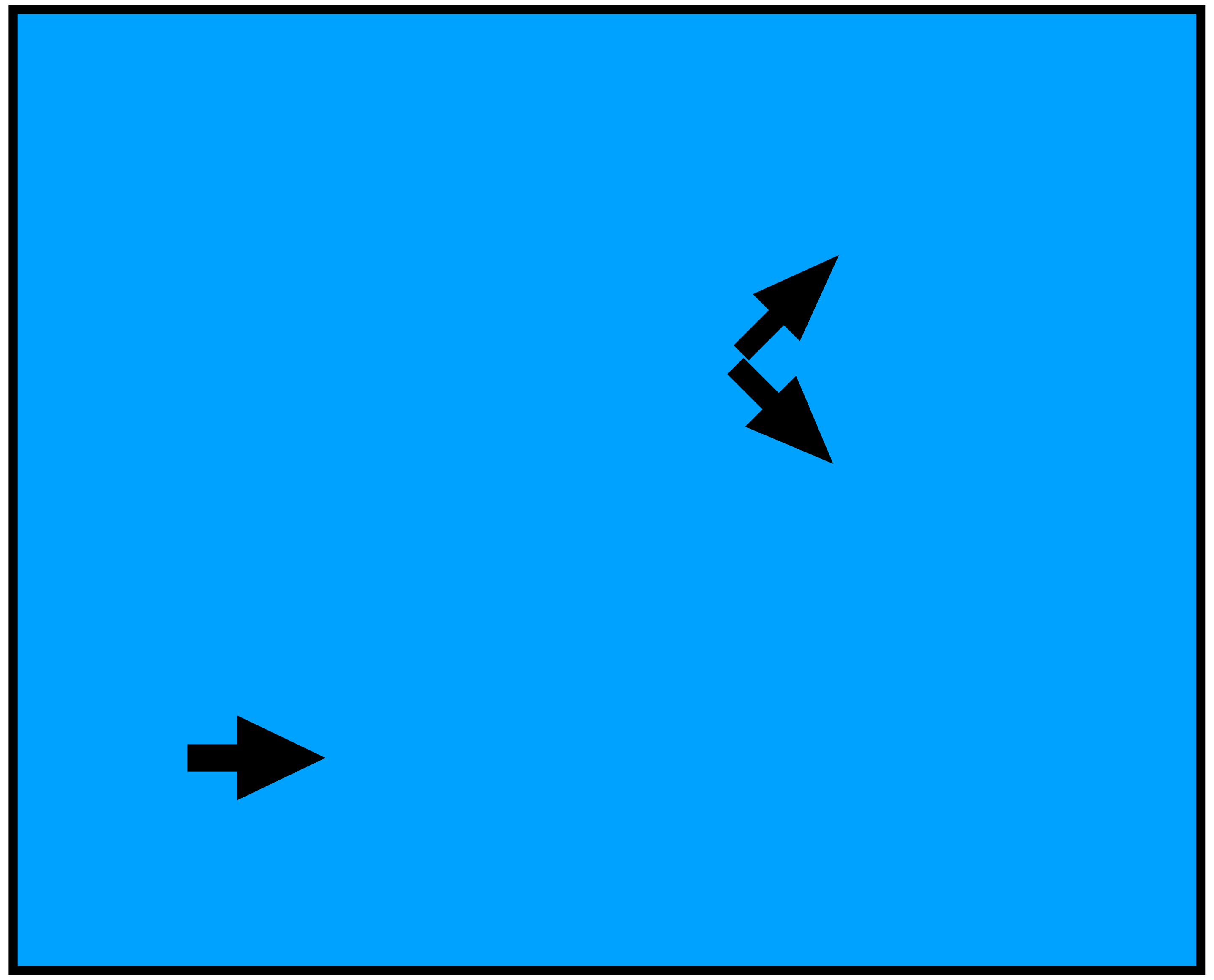

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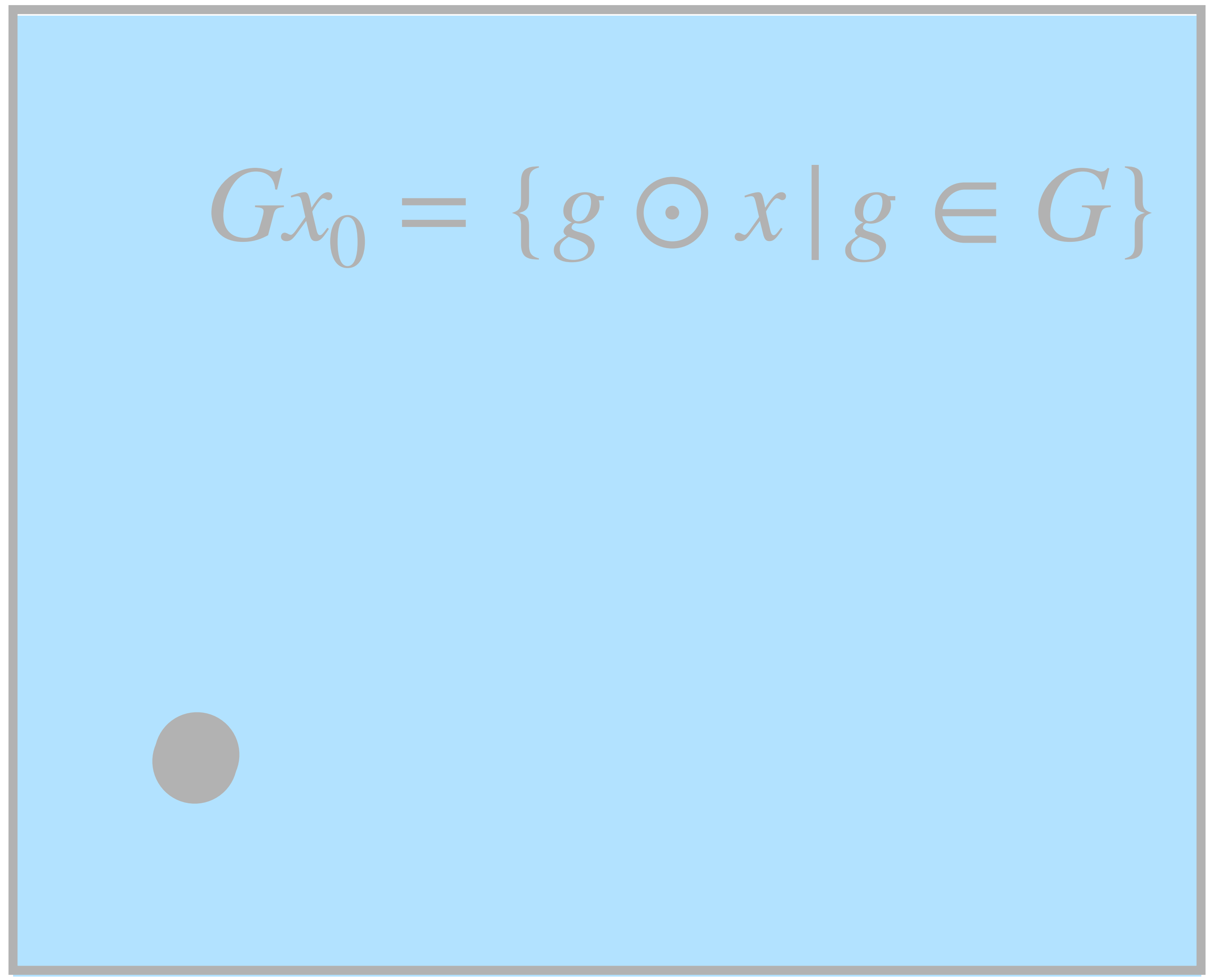

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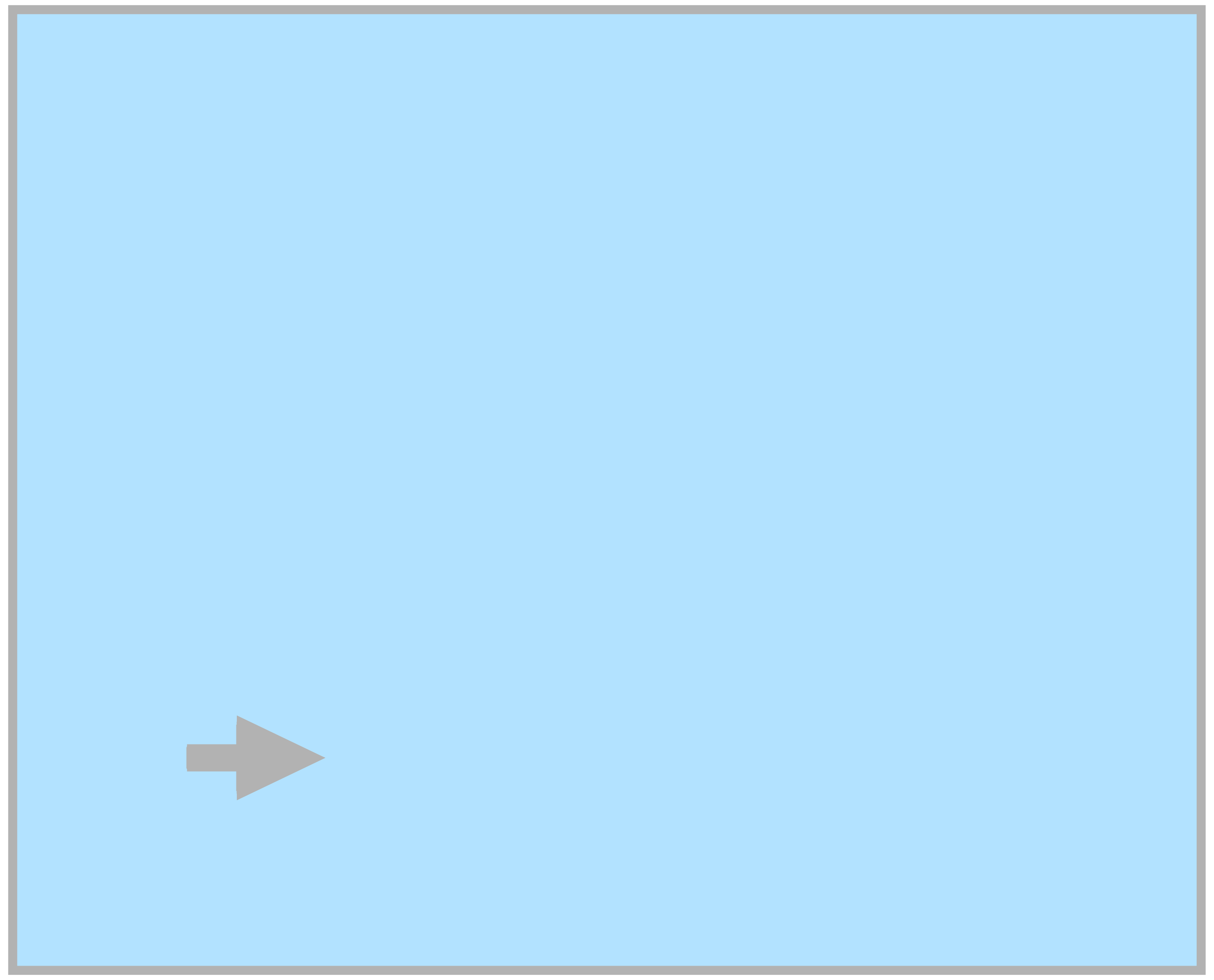

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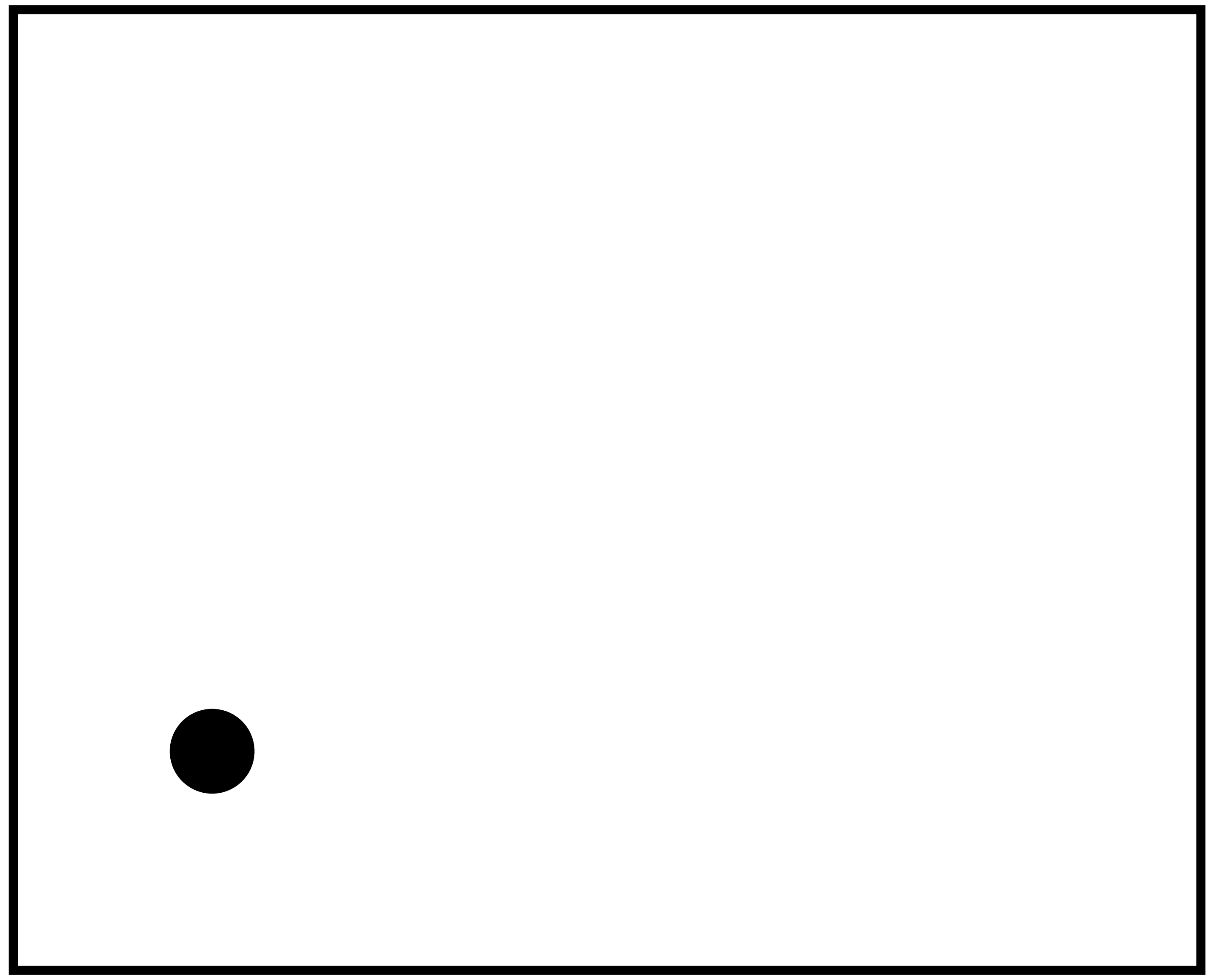

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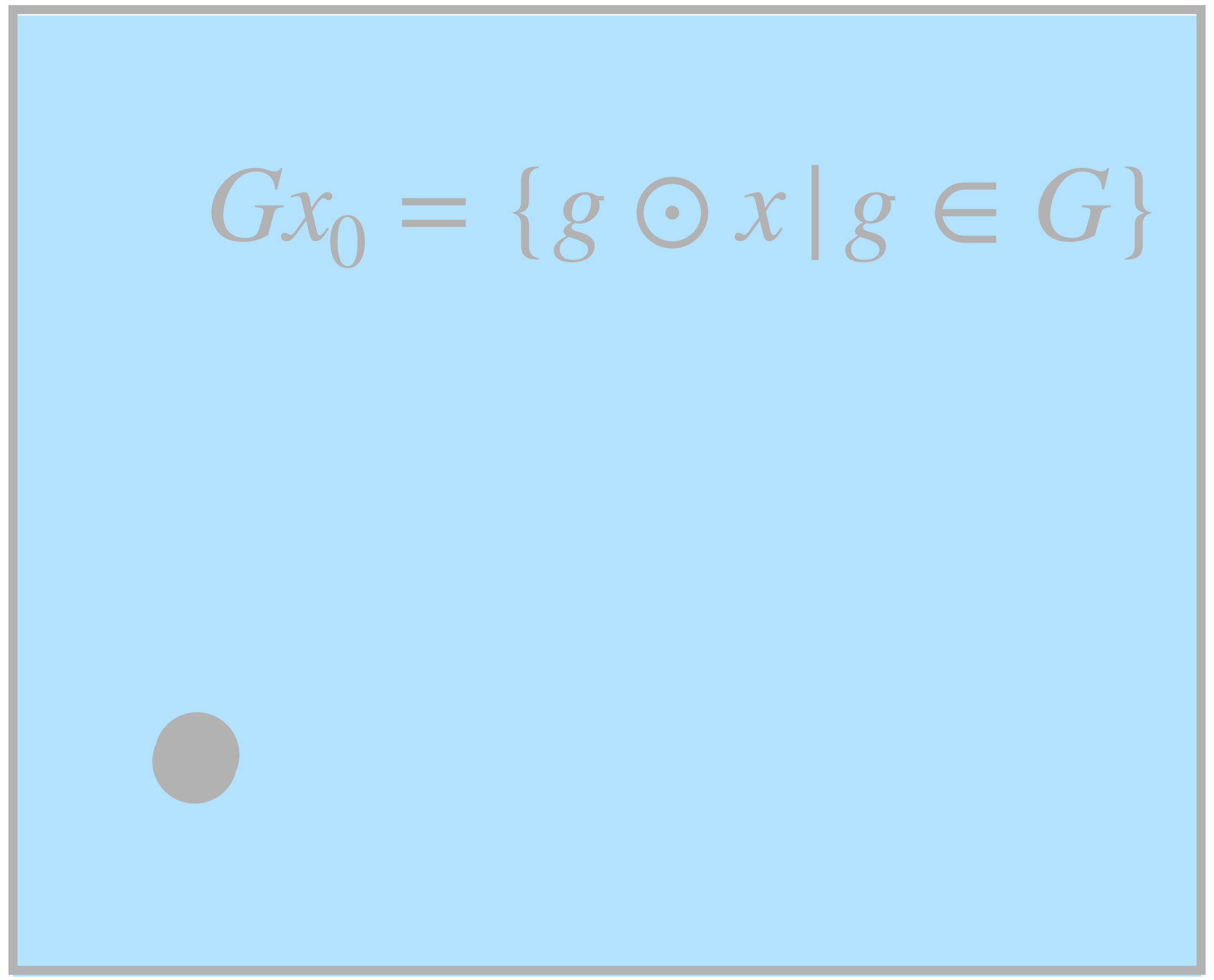

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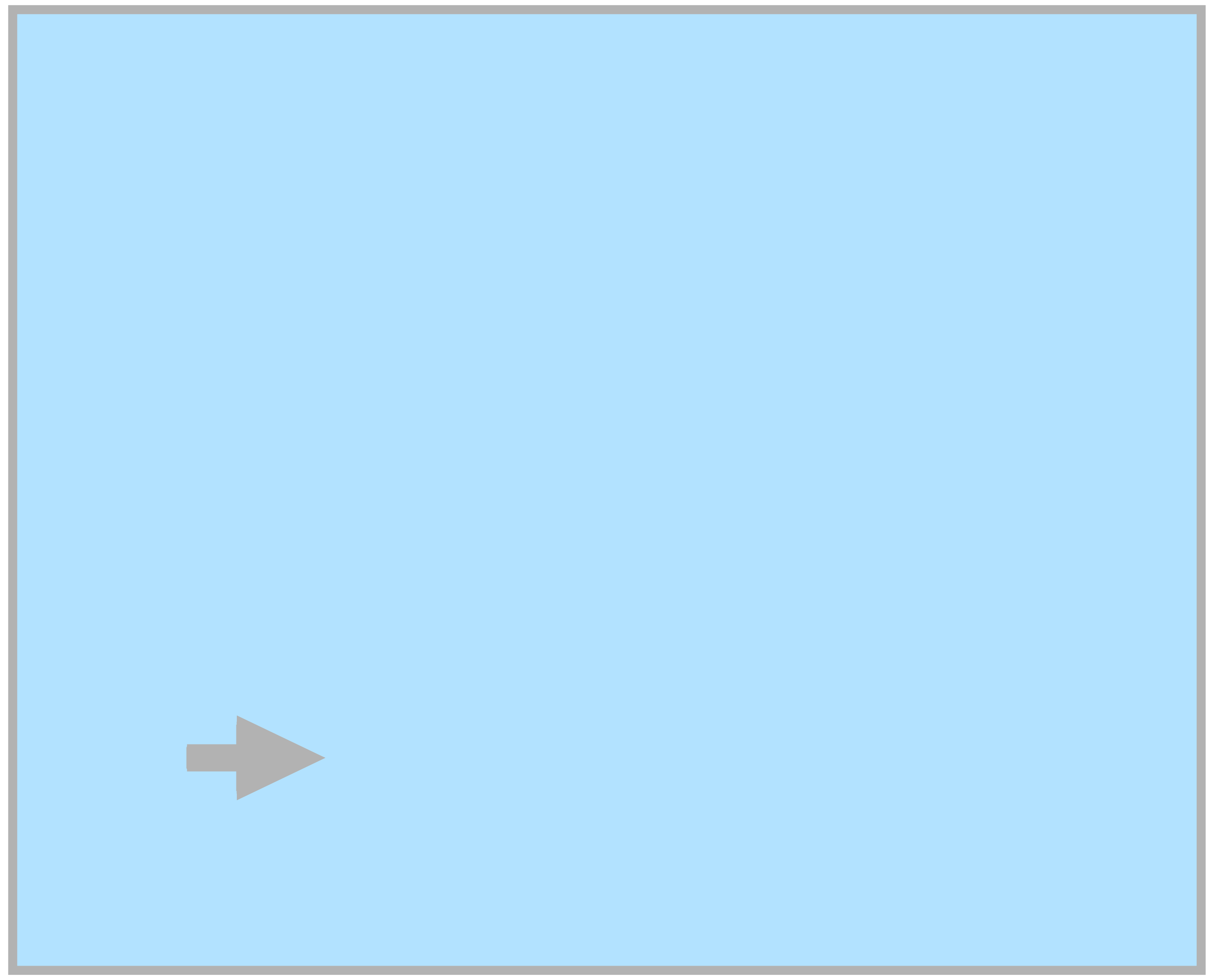

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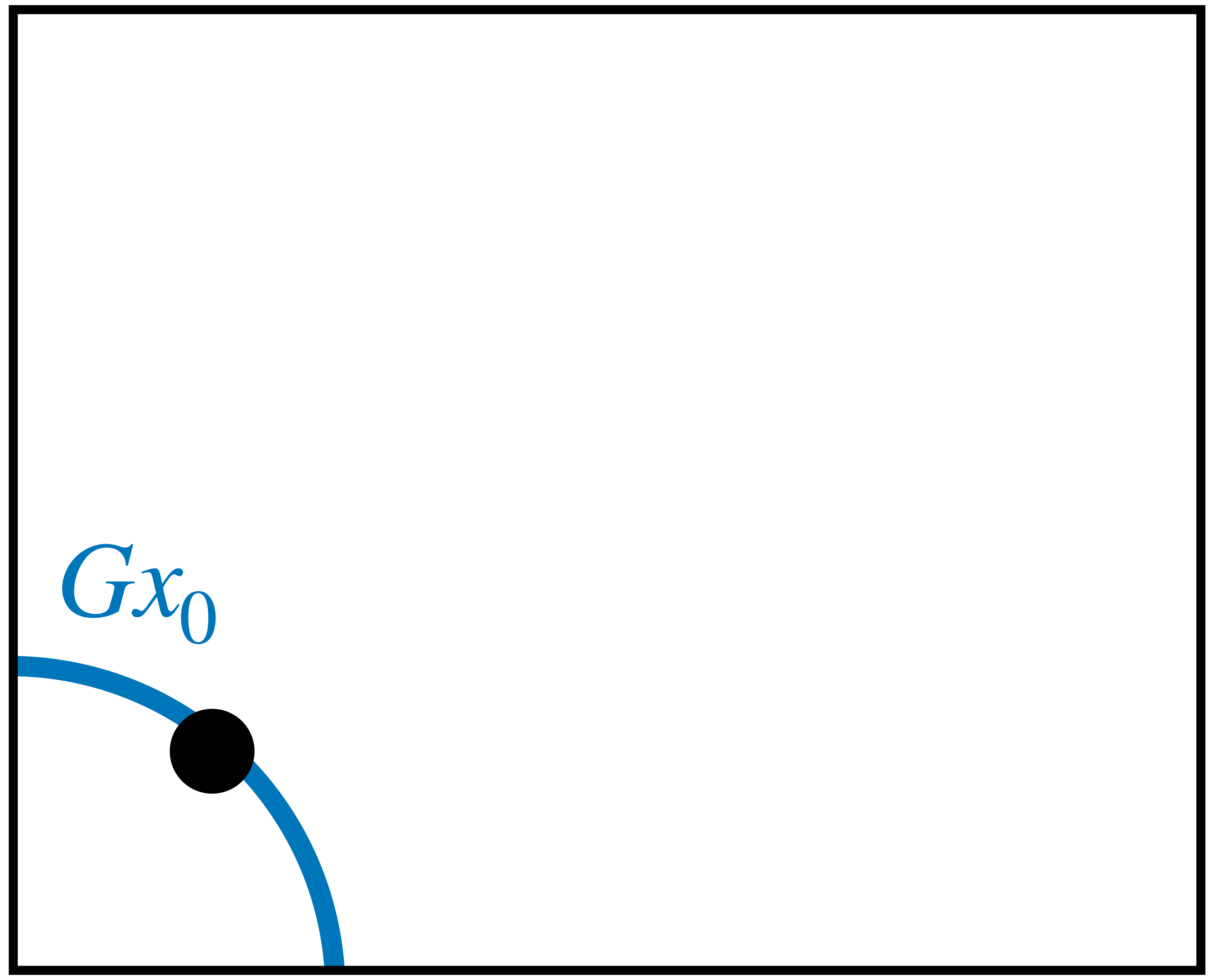

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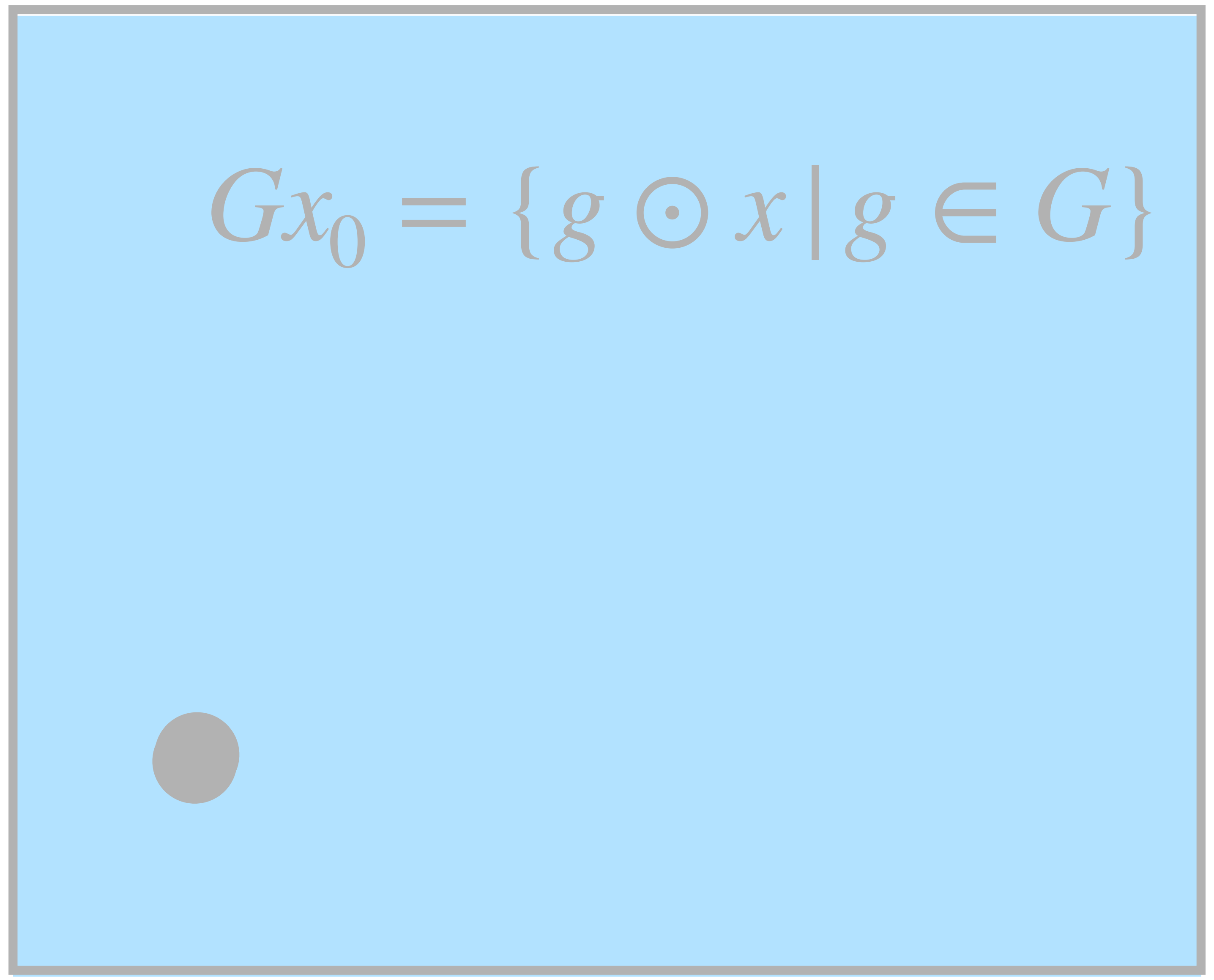

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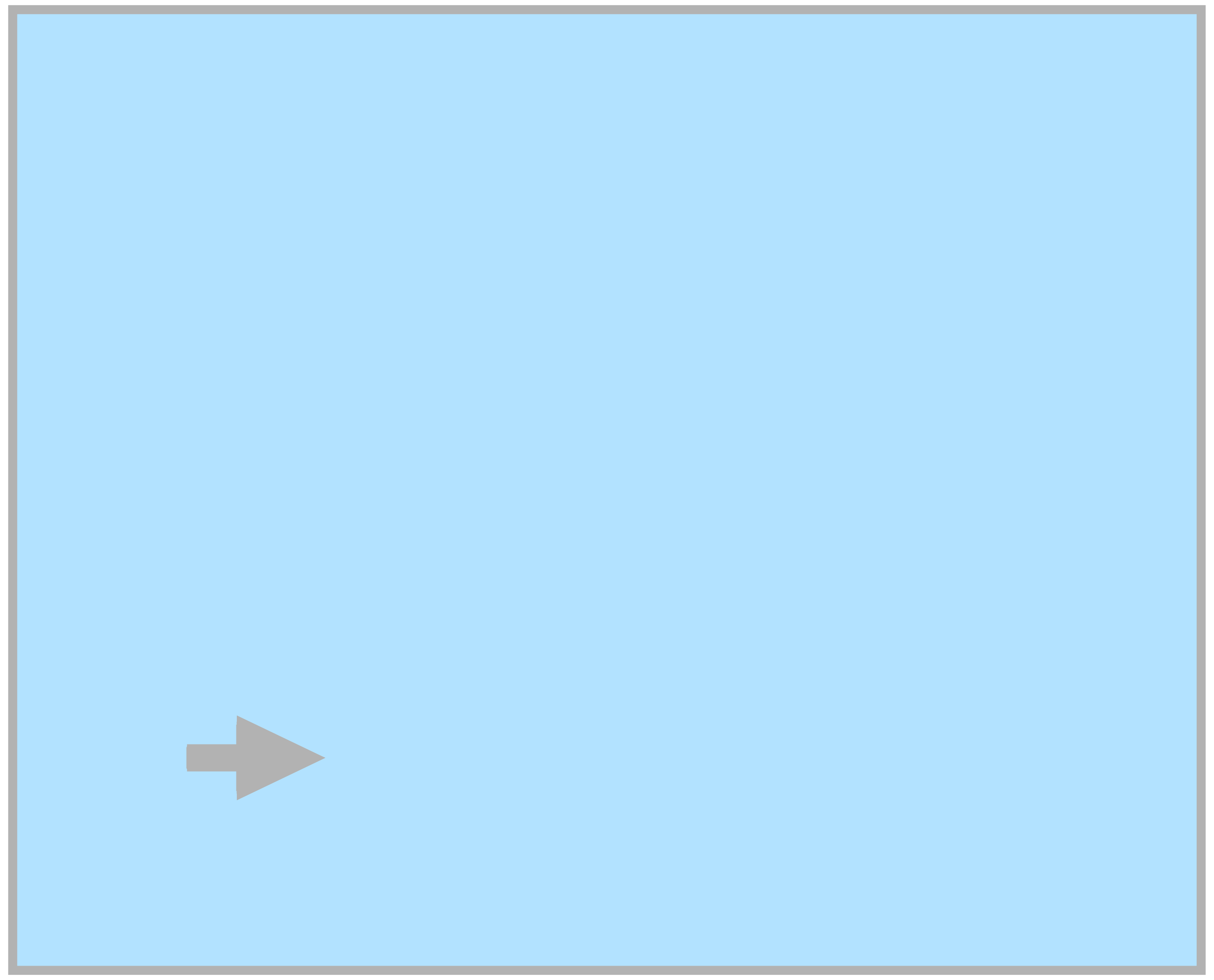

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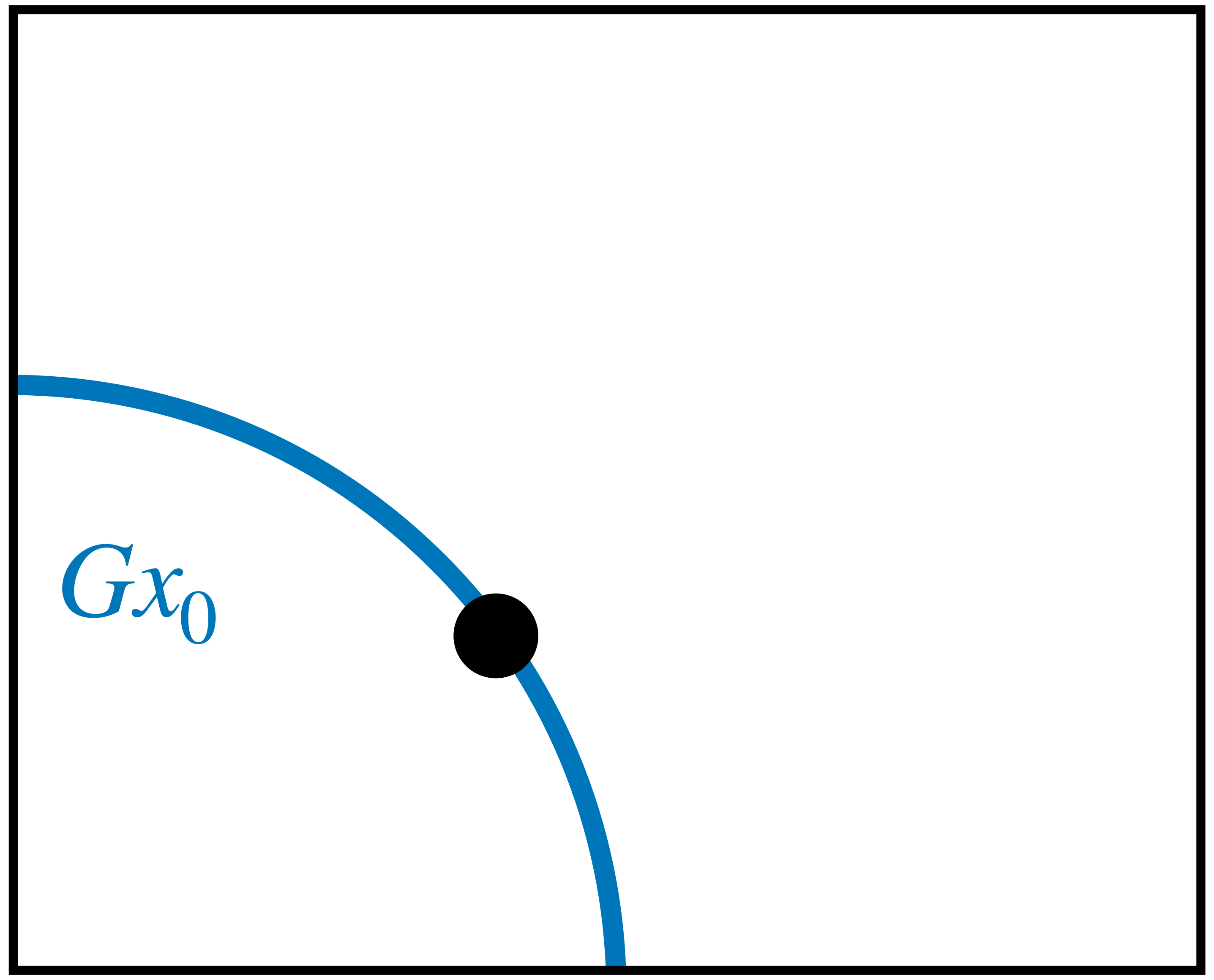

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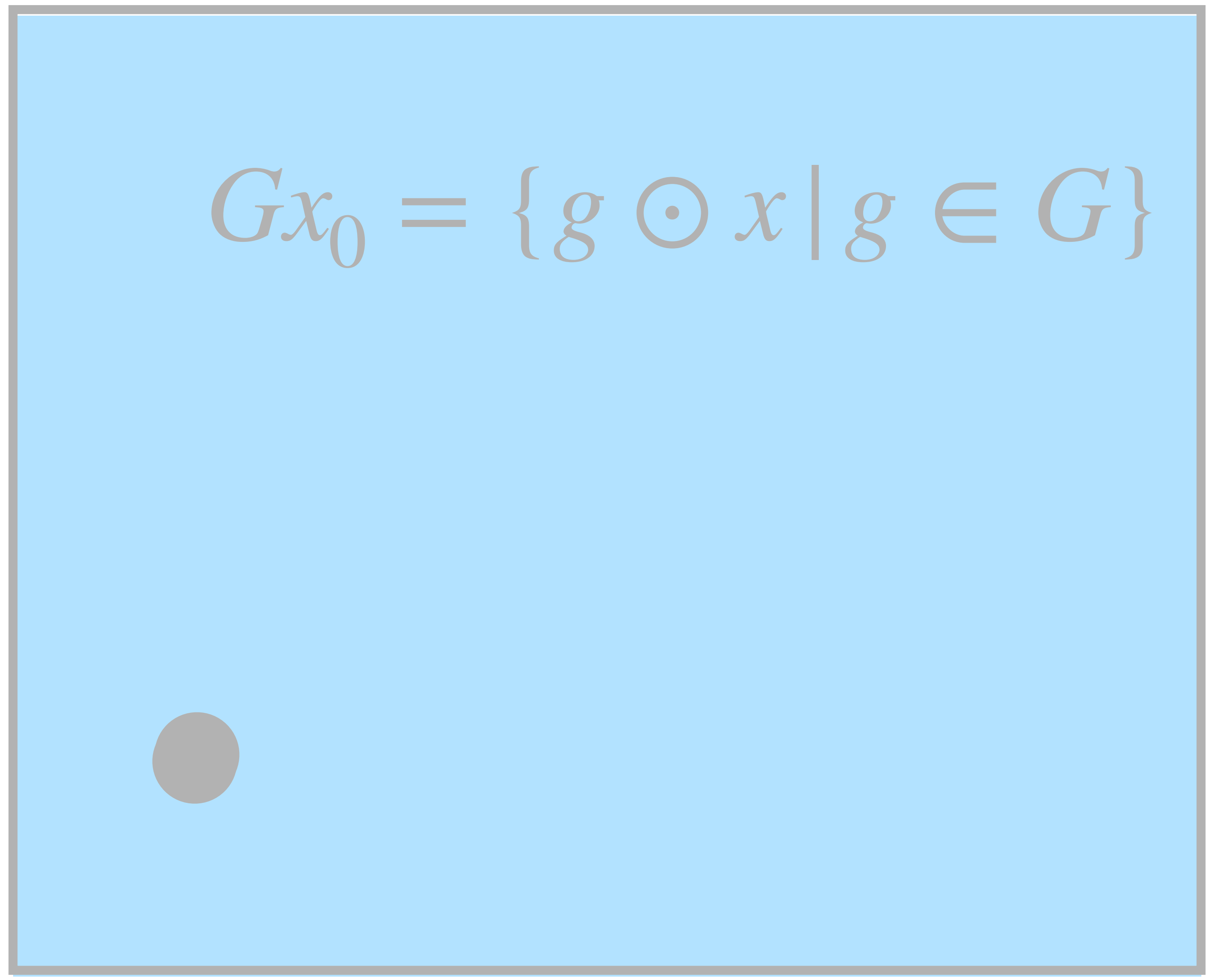

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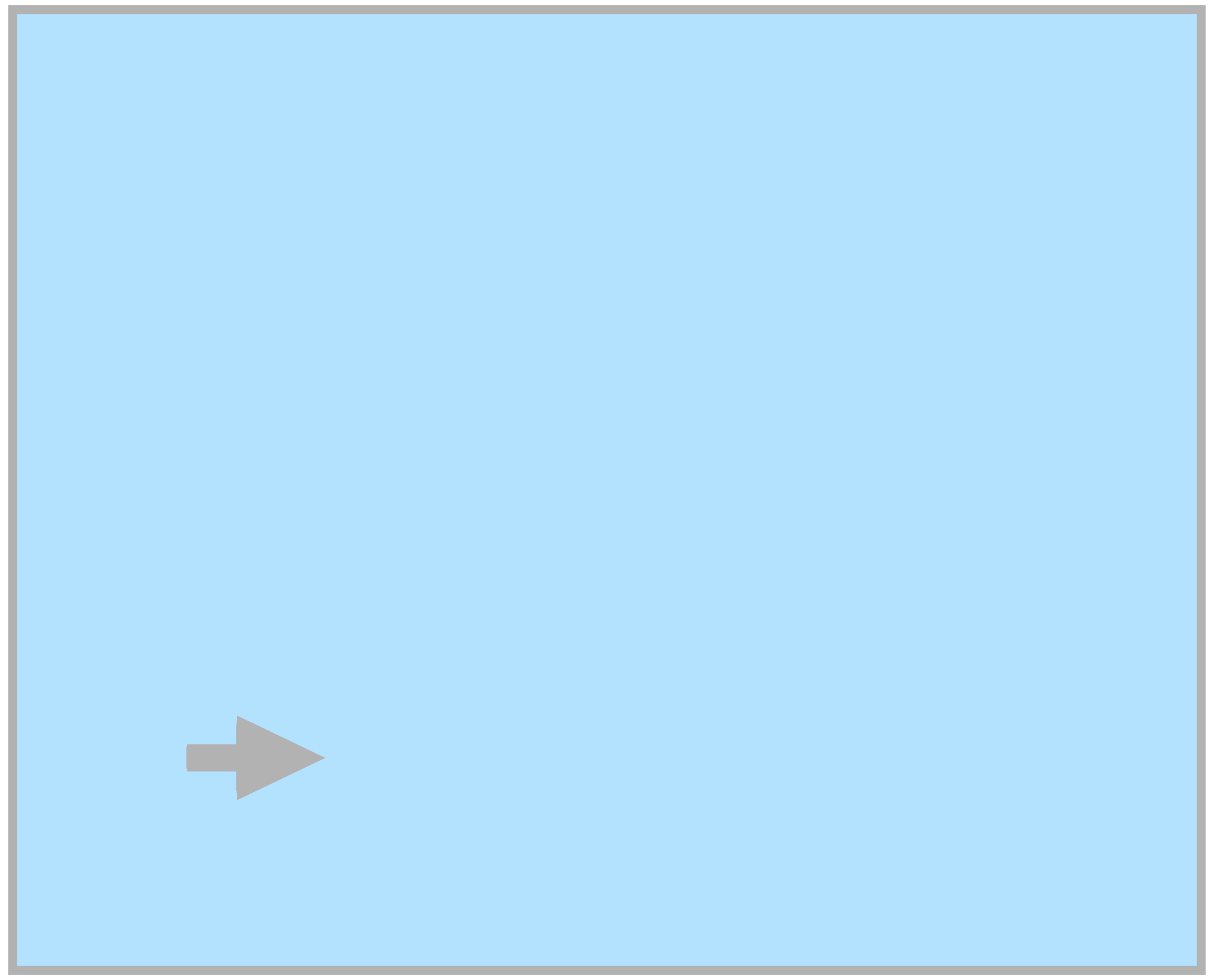

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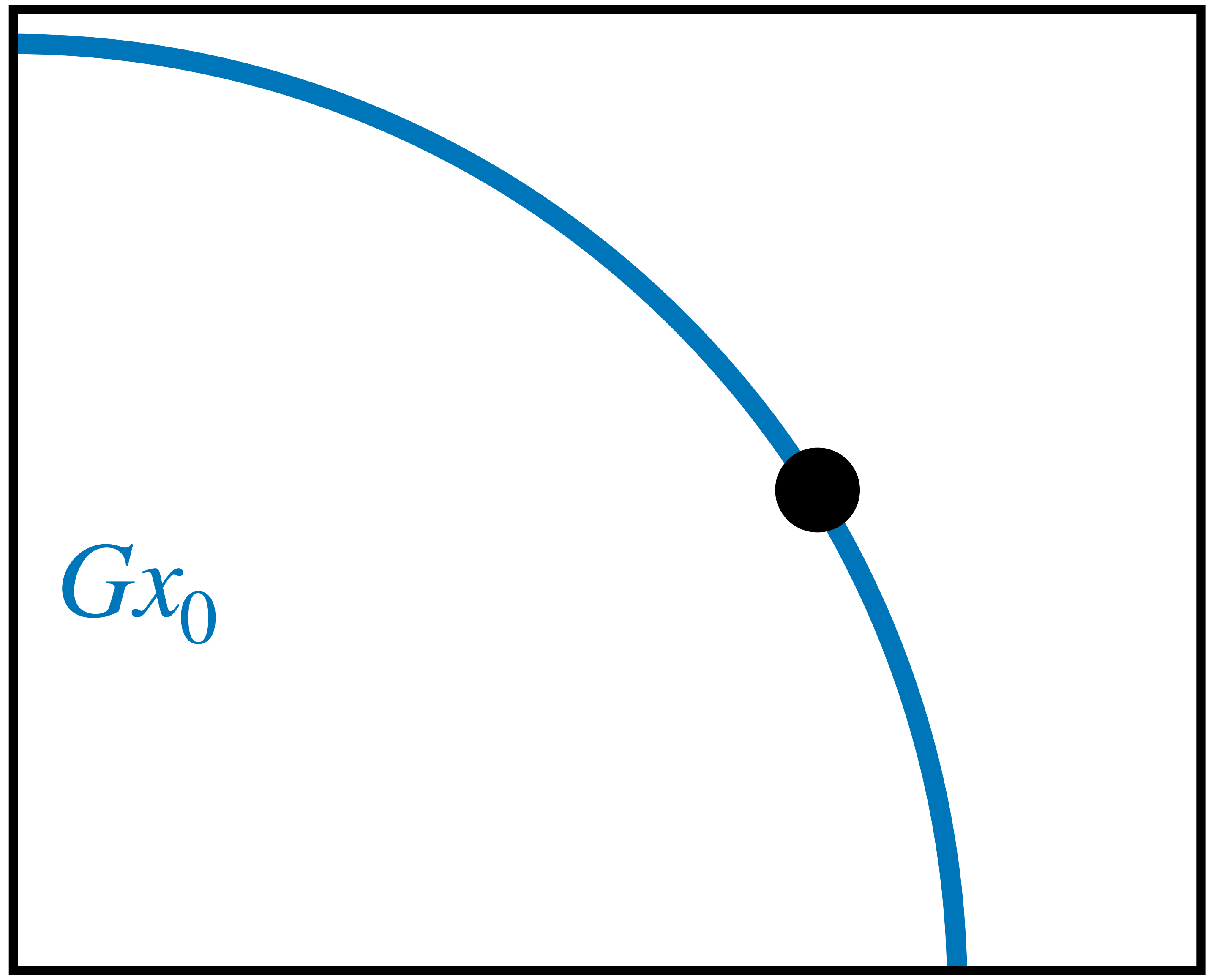

Homogeneous space

Homogeneous space: A space X on which G acts transitively.

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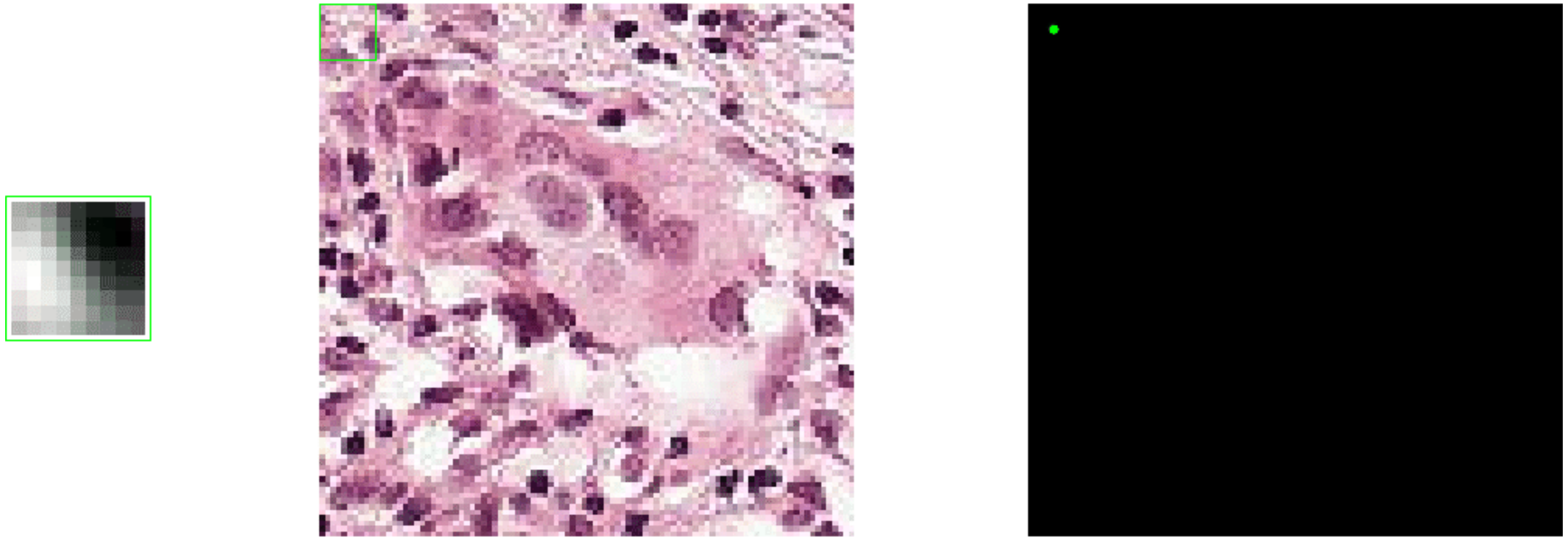
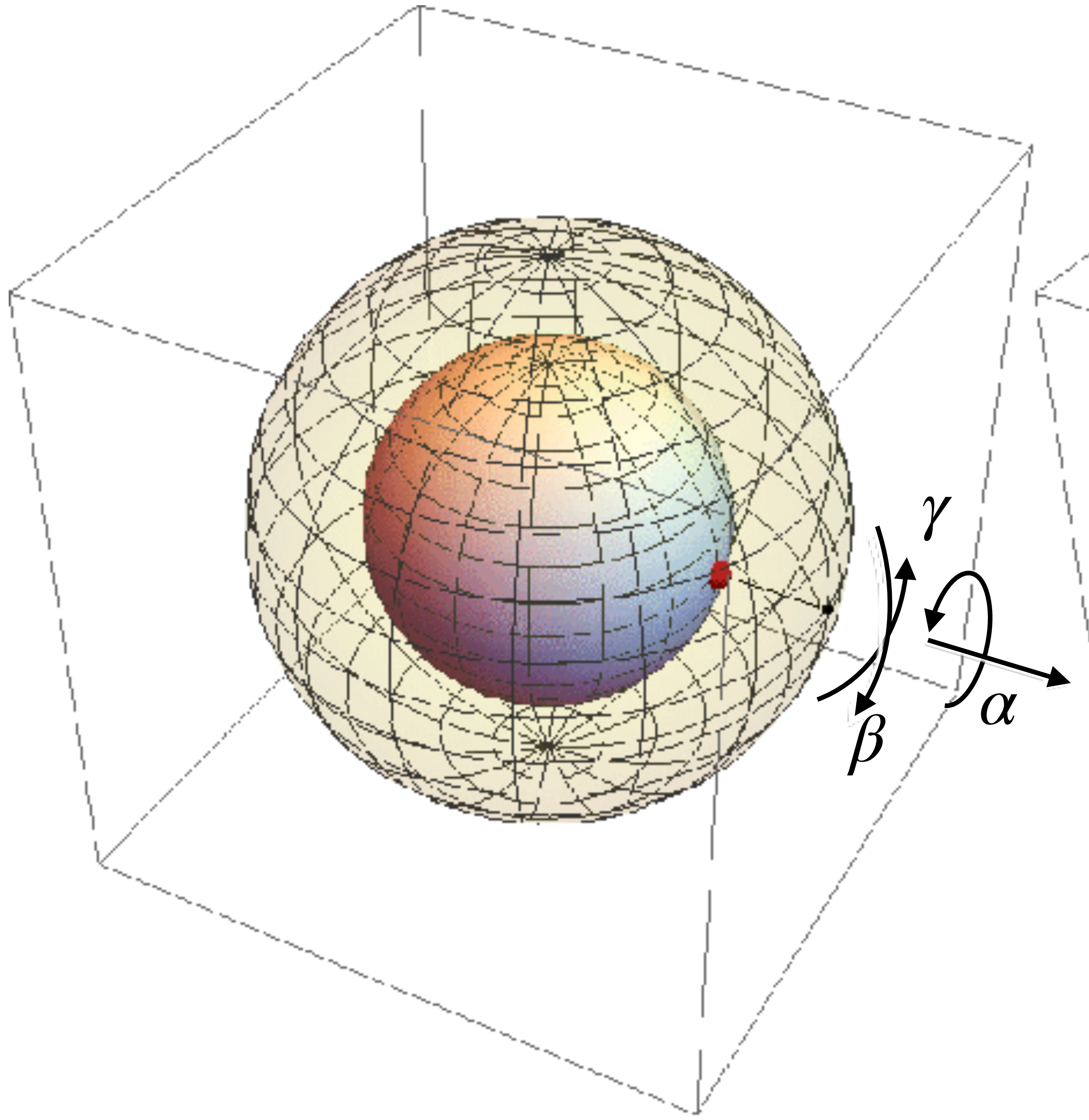


Homogeneous space: the sphere S^2

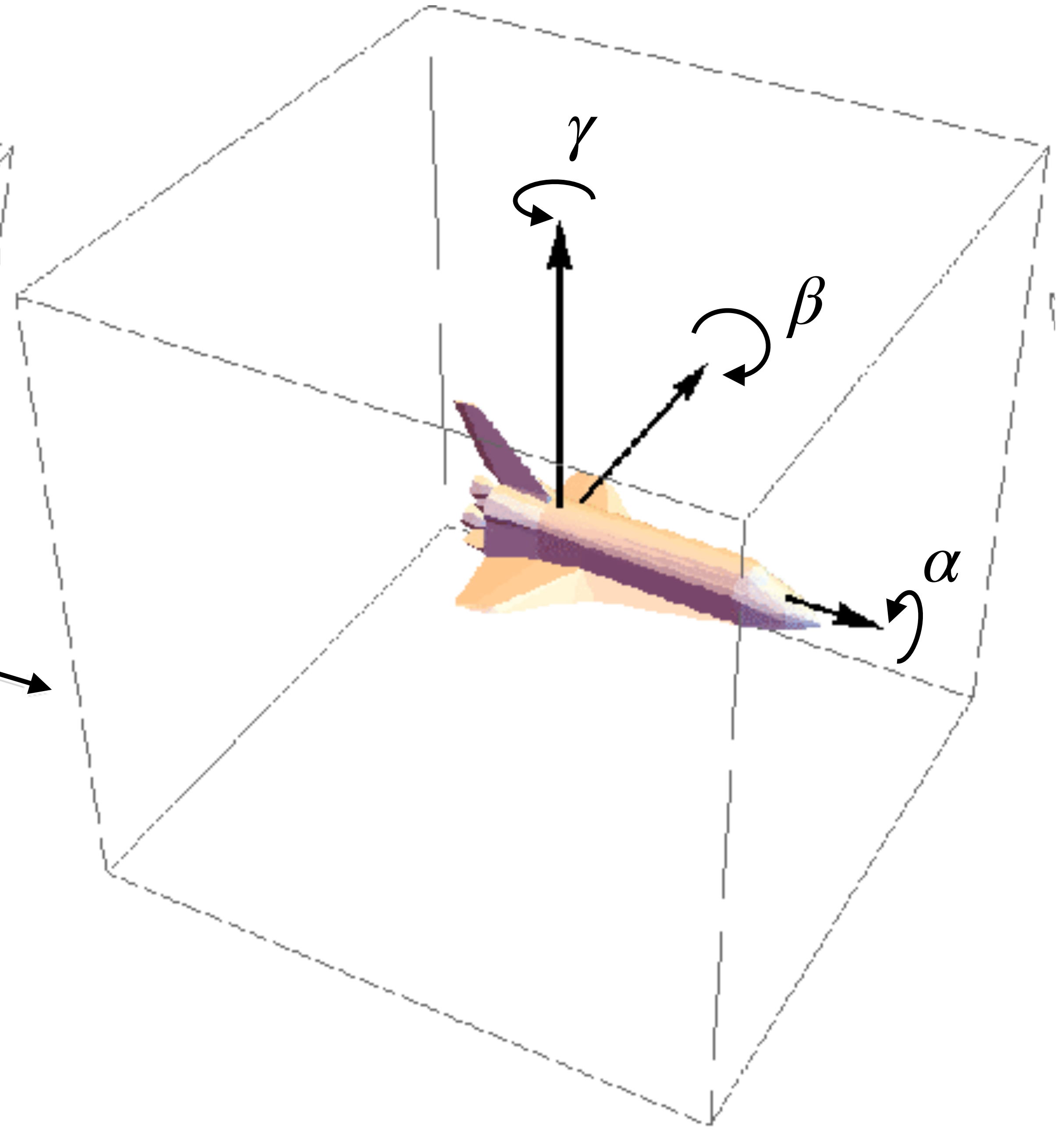
The sphere S^2 is a homogeneous space of 3D rotations $SO(3)$

The 3D rotation group $SO(3)$

Representation in parameter space (XYZ-Euler angles)



Rotation by $R \in SO(3)$
$R = R_{e_z,\gamma} R_{e_y,\beta} R_{e_x,\alpha}$



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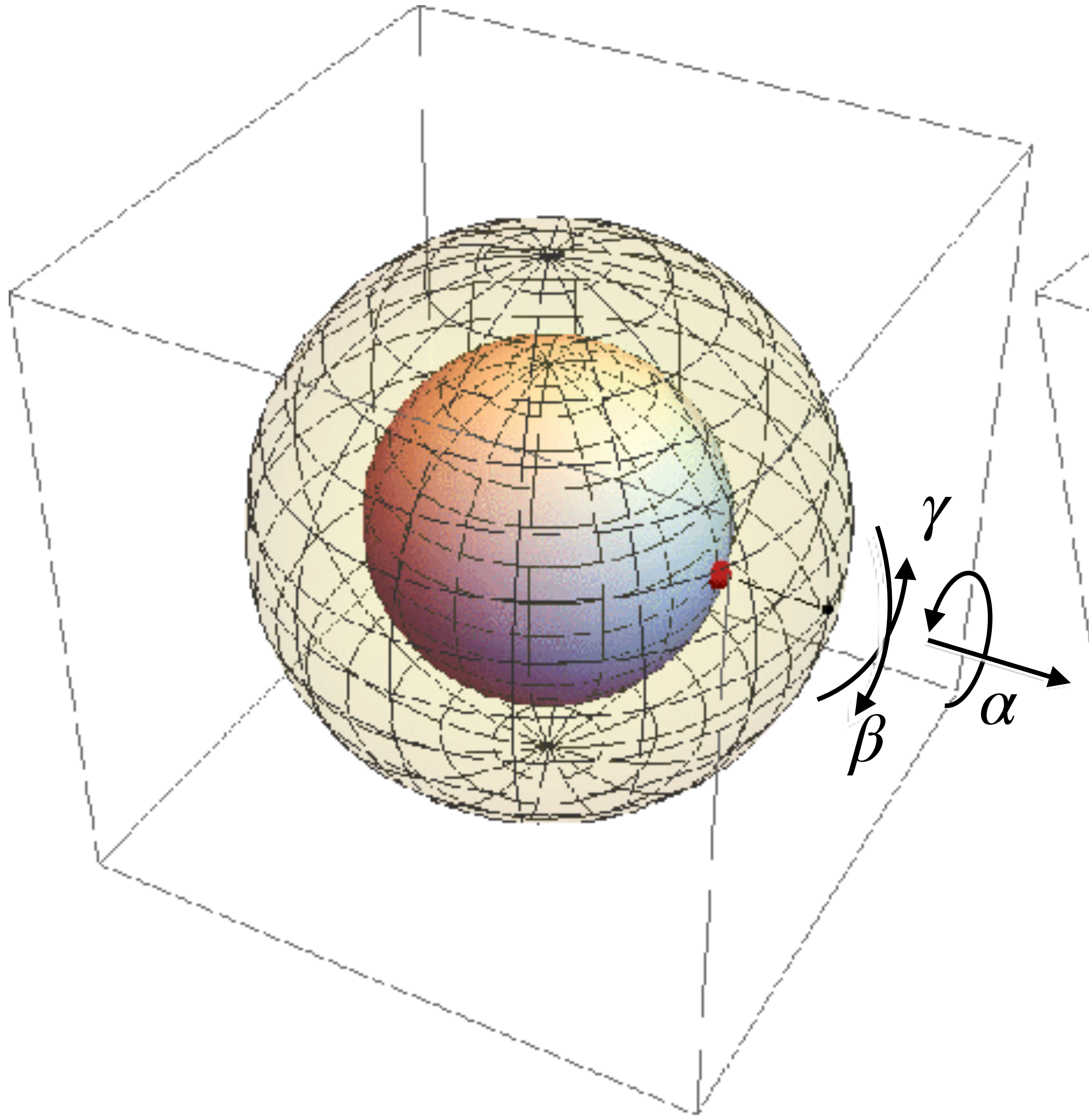

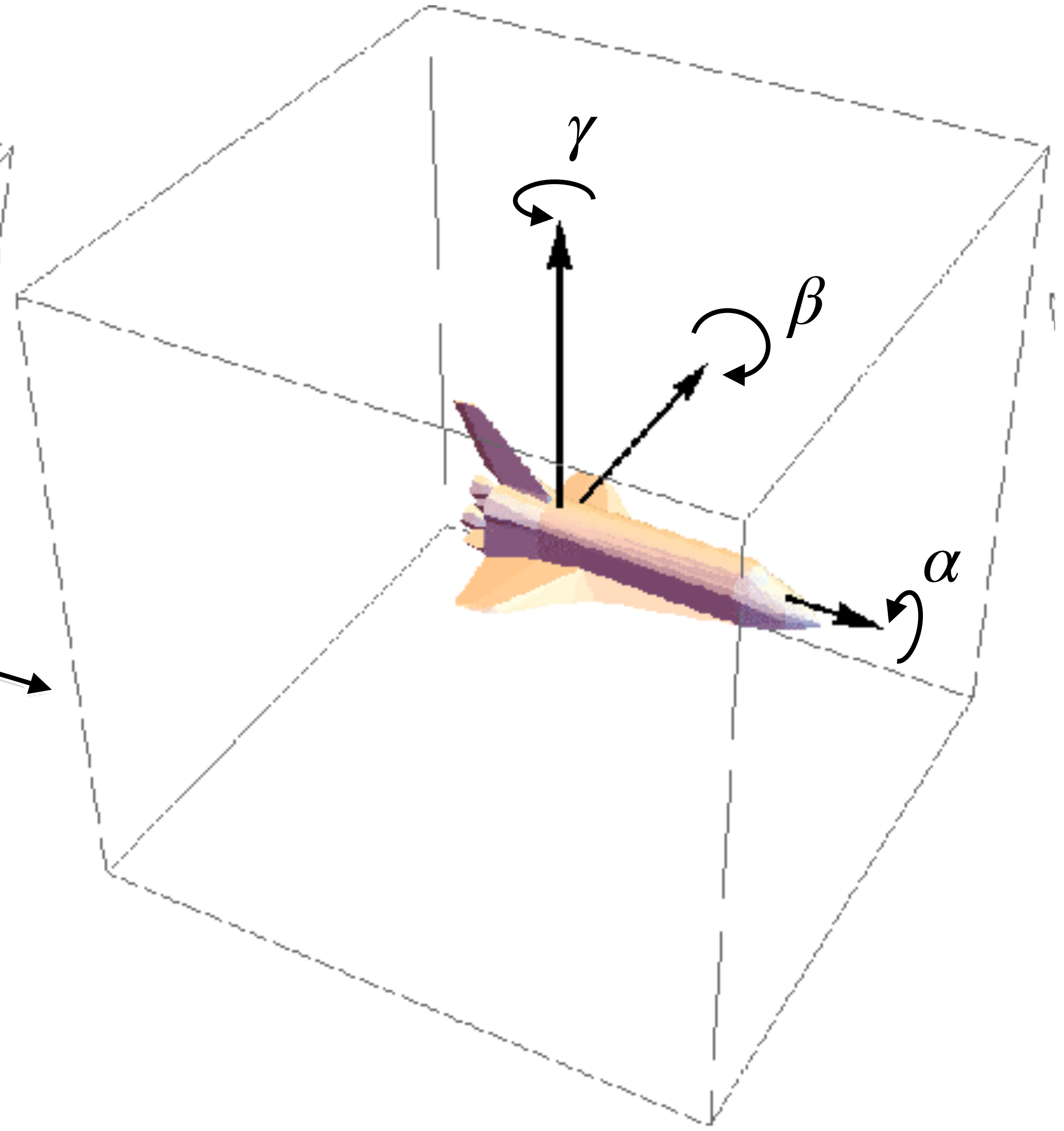

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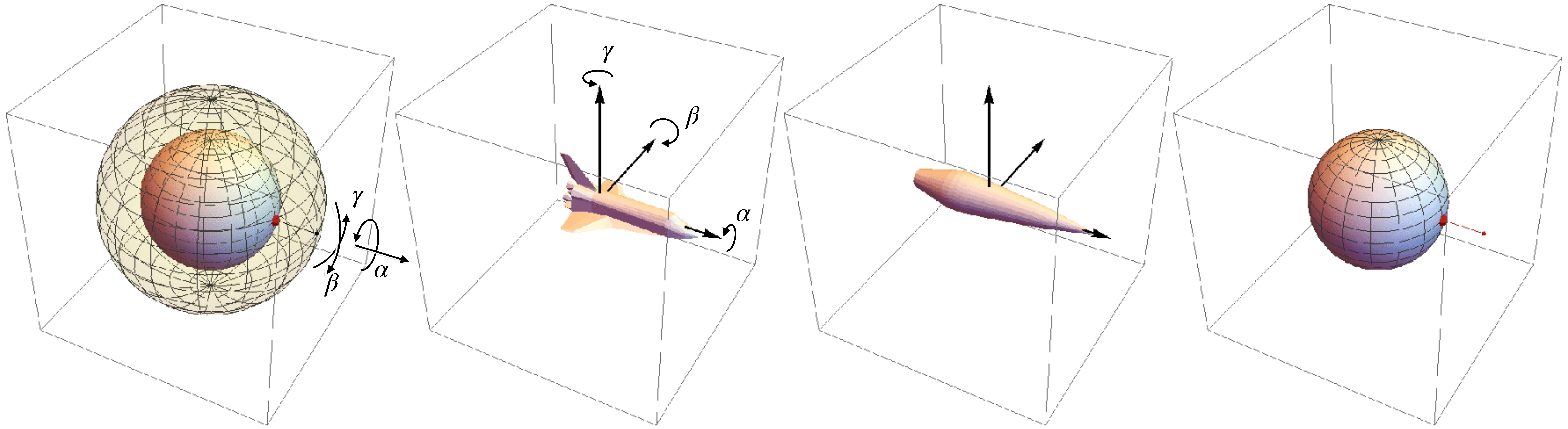
The 2-sphere as a quotient space

Representation in parameter space (XYZ-Euler angles)

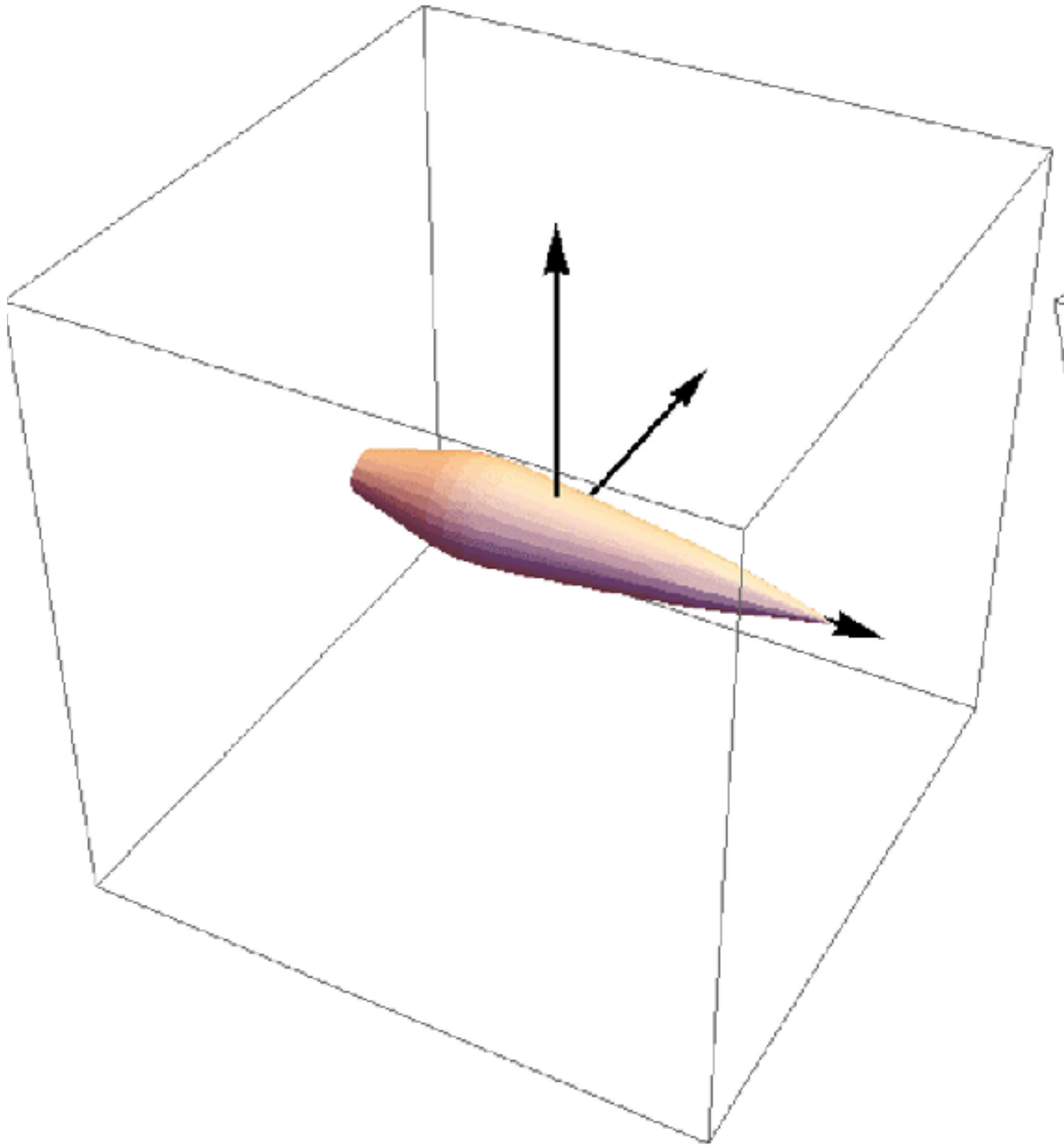
Rotation by $R \in SO(3)$
$R = R_{e_z,\gamma} R_{e_y,\beta} R_{e_x,\alpha}$

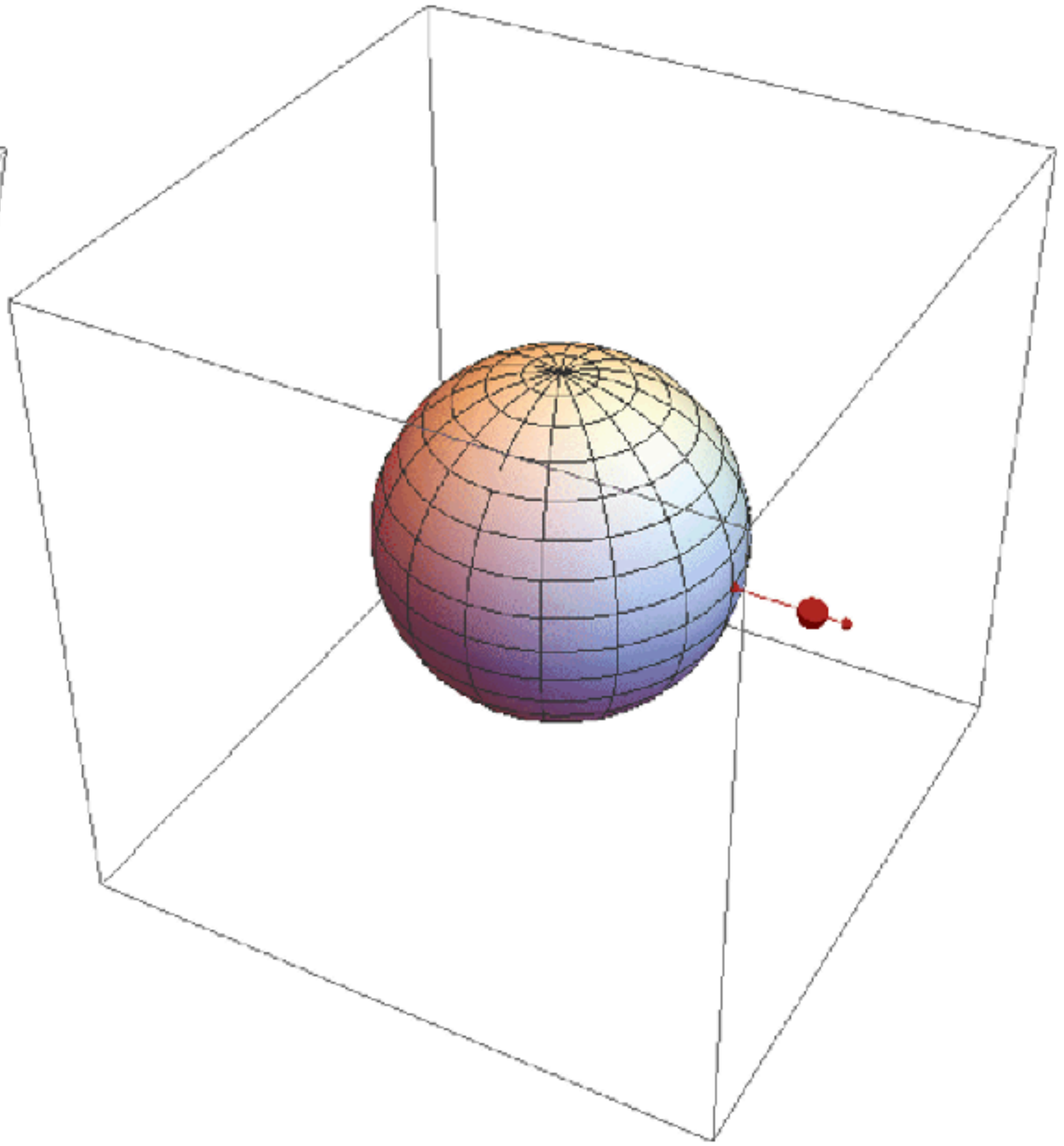
$S^2 \equiv SO(3)/SO(2)$

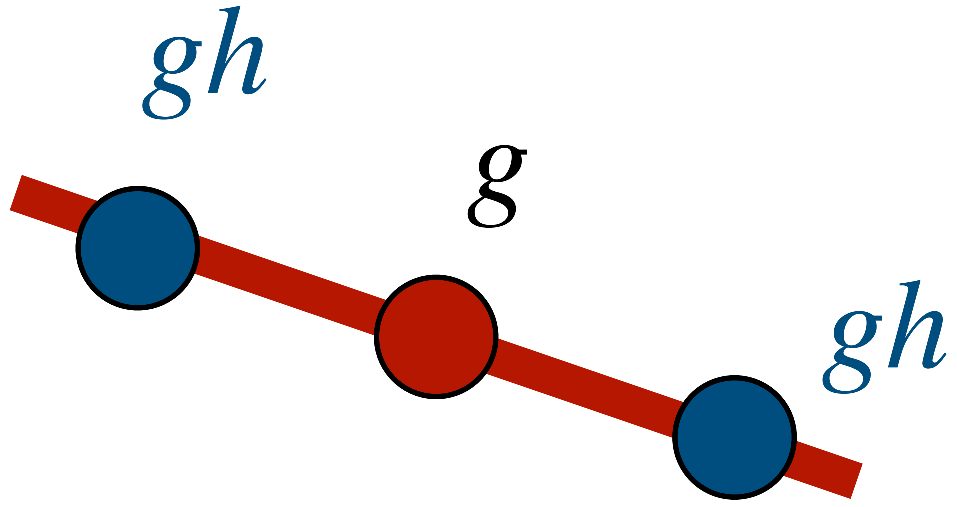


Quotient space

Quotient space G/H: The space of unique cosets $gH = \{gh \,|\, h \in H\}$. Elements of the space G/H are cosets.





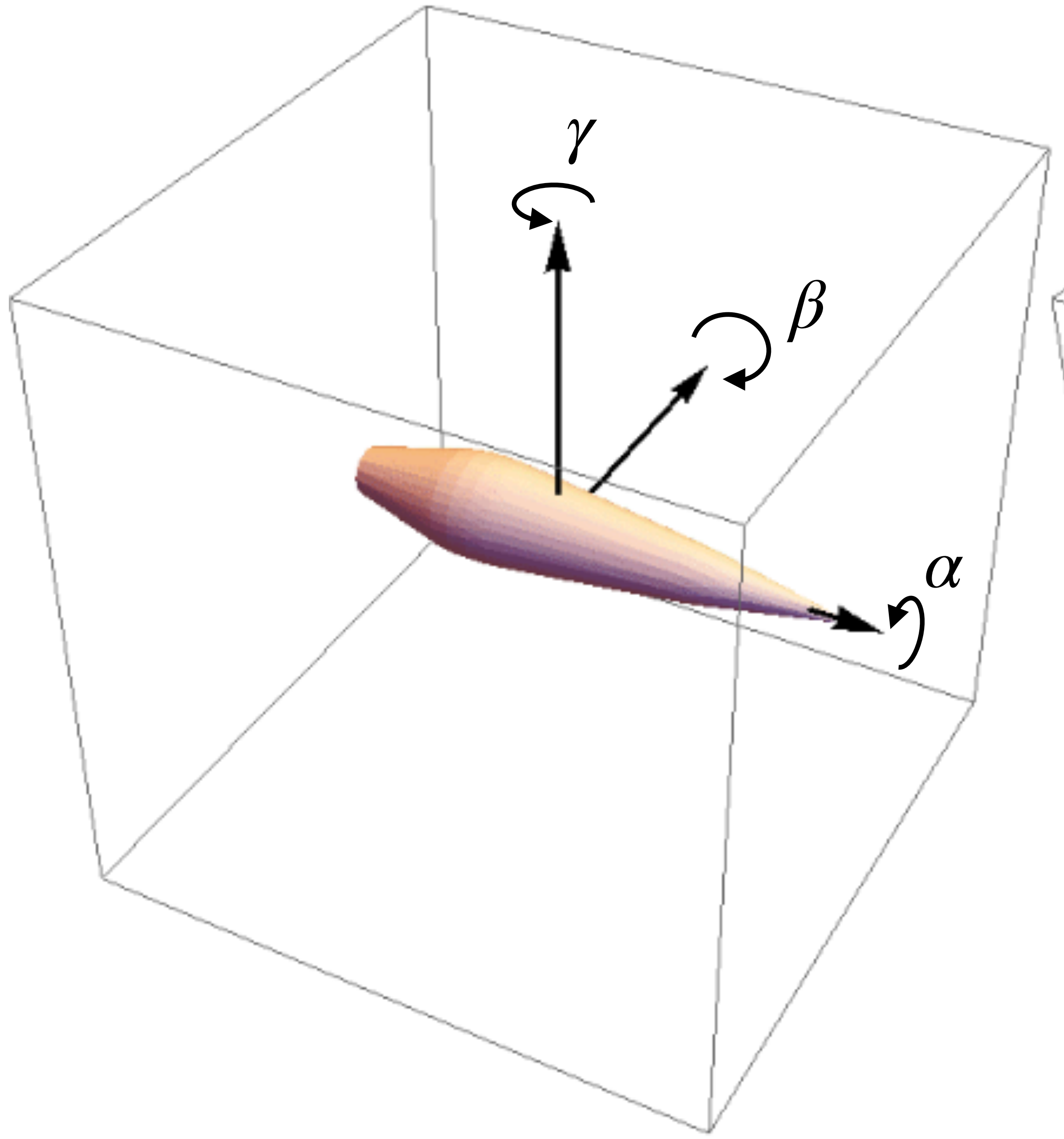


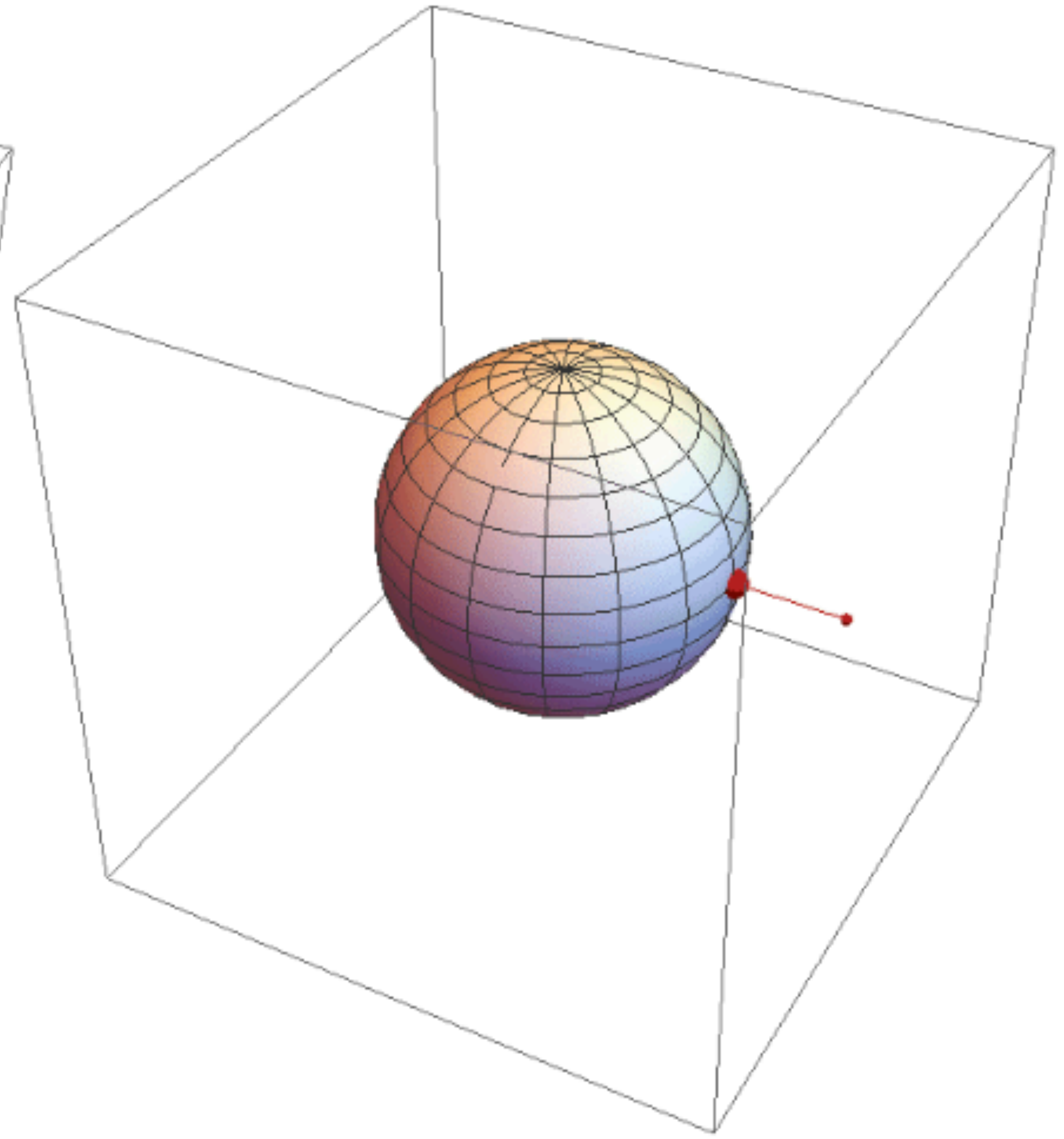
$$gH = \{gh \,|\, h \in H\}$$

Stabilizer

Stabilizer: $\mathrm{Stab}_G(x_0)$ is a subset of G that leaves x_0 unchanged. I.e.,
$\mathrm{Stab}_G(x_0) = \{g \mid gx_0 = x_0\}$

Rotation by $R \in SO(3)$
$R = R_{e_z,\gamma} R_{e_y,\beta} \boxed{R_{e_x,\alpha}}$





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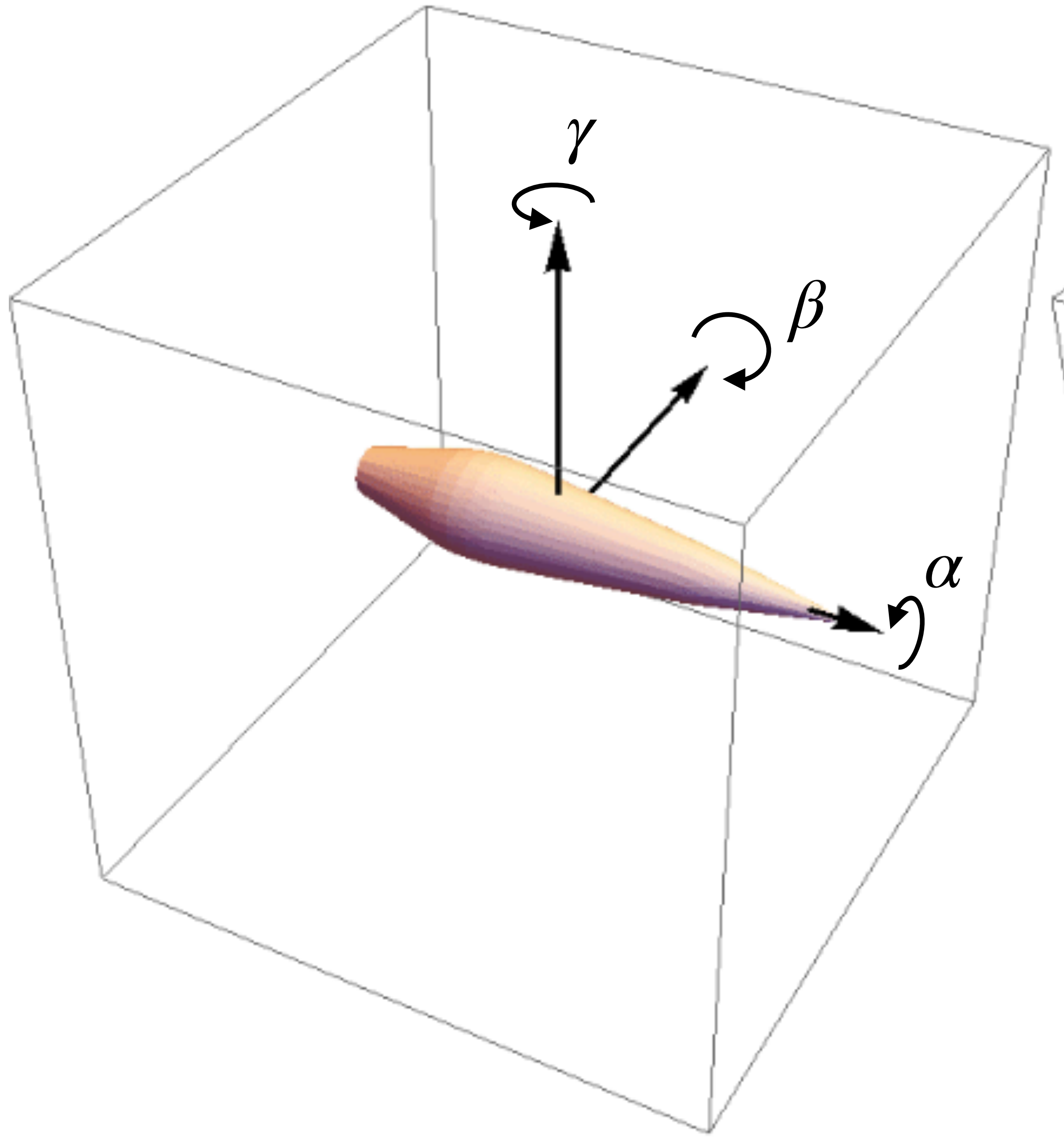

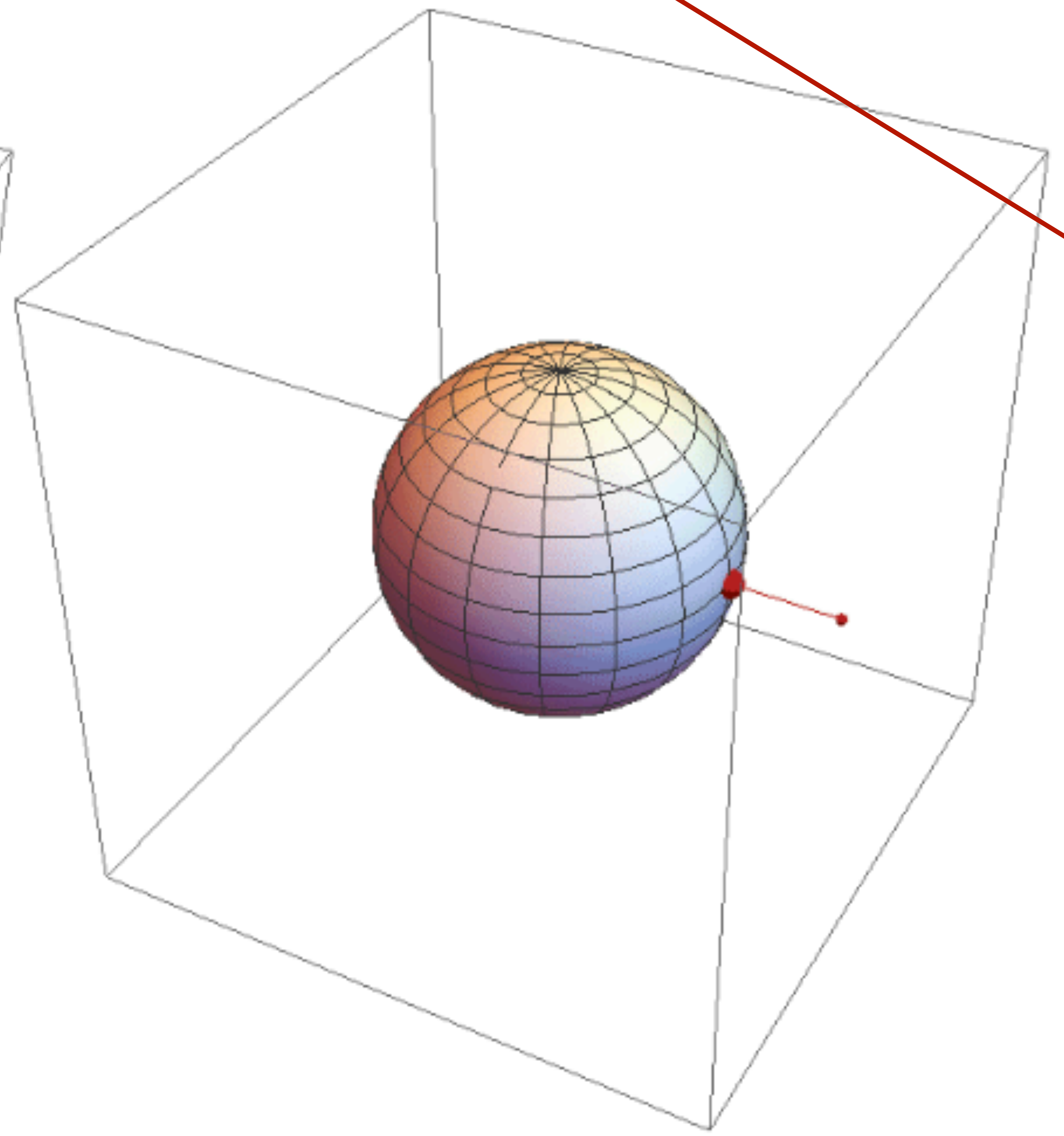

So the sphere is a quotient space

$$S^2 \equiv SO(3)/H$$

with

$$H = \mathrm{Stab}_{SO(3)}(\mathbf{e}_x)$$

Homogeneous space $\equiv$ Quotient space

Any quotient space is a homogeneous space

Any homogeneous space is a quotient space

Quotient space: the Euclidean plane $\mathbb{R}^d$

Lecture notes

Example 2.7 (Quotient space $\mathbb{R}^d = SE(d)/SO(d)$). Let $H = (\{\mathbf{0}\} \times SO(d)$ the subgroup of rotations in $SE(d)$, with $\mathbf{0}$ the identity element of the translationg roup $(\mathbb{R}^d, +)$. The the cosets gH are given by

$$
\begin{aligned}
gH &= \{g \cdot (\mathbf{0}, \tilde{\mathbf{R}}) \mid \tilde{\mathbf{R}} \in SO(d)\} \\
&= \{(\mathbf{R}\mathbf{e} + \mathbf{x}, \mathbf{R}\tilde{\mathbf{R}}) \mid h \in SO(d)\} \\
&= \{(\mathbf{x}, \mathbf{R}\tilde{\mathbf{R}}) \mid \tilde{\mathbf{R}} \in SO(d)\} \\
&= \{(\mathbf{x}, \tilde{\mathbf{R}}) \mid \tilde{\mathbf{R}} \in SO(d)\} \,,
\end{aligned}
$$

with $g = (\mathbf{x}, \mathbf{R})$. So, the cosets are given by all possible rotations for a fixed translation vector $\mathbf{x}$, the vector $\mathbf{x}$ thus indexes these sets. We can therefore make the identification

$$\mathbb{R}^d \equiv SE(d)/SO(d)\,.$$

We already saw in Exercise 2.1 that $\mathbb{R}^d$ is a homogeneous space of $SE(d)$, this is a consequence of Lemma 2.1.

Conclusion

- A homogeneous space X is a space on which a group G acts transitively
- This is important as then we can reach any point in X via the action of G (when template matching we want to scan the entire space)
- Any homogeneous X space can be identified with a quotient space $X \equiv G/H$ (there is a point that is left invariant by a sub-group H)