Group Equivariant Deep Learning

Lecture 1 - Regular group convolutions

Lecture 1.3 - Regular group convolutions | Template matching viewpoint

General group convolutional NN design with example for roto-translation equivariance (SE(2))

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This mini-course serves as a module with the UvA Master AI course Deep Learning 2: [https://uvadl2c.github.io/](https://uvadl2c.github.io/)
Cross-correlations

\[(k \star_{\mathbb{R}^2} f)(x) = \int_{\mathbb{R}^2} k(x' - x)f(x')dx'\]
Cross-correlations

\[(k \star_{\mathbb{R}^2} f)(x) = \int_{\mathbb{R}^2} k(x' - x)f(x')dx' = (\mathcal{L}_g k, f)_{L^2(\mathbb{R}^2)}\]

Are convolutions with reflected conv kernels (and vice versa)

Representation of the translation group!
Cross-correlations

\[(k \star_{\mathbb{R}^2} f)(x) = \int_{\mathbb{R}^2} k(x' - x)f(x')dx' = (\mathcal{L}_g k, f)_{L^2(\mathbb{R}^2)}\]

Are convolutions with reflected conv kernels (and vice versa)

2D convolution kernel

2D feature map

2D feature map (after ReLU)
Equivariance

Convolutions/cross-correlations are translation equivariant

$$\Phi$$

$$\mathcal{L}_{\mathbb{R}^2 \to \mathbb{L}_2(\mathbb{R}^2)}(x)$$

$$\Phi$$

$$(k \ast_{\mathbb{R}^2} f)(x) = (\mathcal{L}_{\mathbb{R}^2 \to \mathbb{L}_2(\mathbb{R}^2)}(x) k, f)_{\mathbb{L}_2(\mathbb{R}^2)}$$

Representation of the translation group
Equivariance

Convolutions are generally not equivariant to roto-translations

$\mathcal{L}_{\theta}^{SO(2)} \rightarrow L^2(\mathbb{R}^2)$

Representation of the rotation group

$(k \ast_{\mathbb{R}^2} f)(x) = (\mathcal{L}_{\mathbb{R}^2} k \circ f)_{L^2(\mathbb{R}^2)}$

Representation of the translation group
Lifting correlations: \((k \star \tilde{f})(x, \theta) = \langle \mathcal{L}_{g}^{SE(2) \to L^2(\mathbb{R}^2)} k, f \rangle_{L^2(\mathbb{R}^2)}\)
\( \text{SE}(2) \) equivariant cross-correlations

**Representation of the roto-translation group!**

Lifting correlations: \((k \tilde{\star} f)(x, \theta) = \left( \mathcal{L}_{g}^{SE(2)\rightarrow \mathbb{L}_{2}(\mathbb{R}^{2})} k, f \right)_{\mathbb{L}_{2}(\mathbb{R}^{2})} = \left( \mathcal{L}_{x}^{\mathbb{R}^{2}\rightarrow \mathbb{L}_{2}(\mathbb{R}^{2})} \mathcal{L}_{\theta}^{SO(2)\rightarrow \mathbb{L}_{2}(\mathbb{R}^{2})} k, f \right)_{\mathbb{L}_{2}(\mathbb{R}^{2})} \)

- \text{translation}
- \text{rotation}
SE(2) equivariant cross-correlations

**Representation of the roto-translation group!**

Lifting correlations: \((k \ast \tilde{f})(x, \theta) = (\mathcal{L}^{SE(2) \rightarrow \mathbb{L}^2(\mathbb{R}^2)}_{g} k \ast f)_{\mathbb{L}^2(\mathbb{R}^2)} = (\mathcal{L}^{\mathbb{R}^2 \rightarrow \mathbb{L}^2(\mathbb{R}^2)}_{x} \mathcal{L}^{SO(2) \rightarrow \mathbb{L}^2(\mathbb{R}^2)}_{\theta} k \ast f)_{\mathbb{L}^2(\mathbb{R}^2)} = k(R^{-1}_\theta(x' - x))\)
SE(2) equivariant cross-correlations

**Representation of the roto-translation group!**

\[
(k \tilde{\star} f)(x, \theta) = (\mathcal{L}^{SE(2)\rightarrow \mathbb{L}^2(\mathbb{R}^2)}_g k \cdot f)_{\mathbb{L}^2(\mathbb{R}^2)} = (\mathcal{L}^{SO(2)\rightarrow \mathbb{L}^2(\mathbb{R}^2)}_\theta k \cdot f)_{\mathbb{L}^2(\mathbb{R}^2)}
\]

**Lifting correlations:**

- Rotated 2D convolution kernel: \( \mathcal{L}^{SO(2)\rightarrow \mathbb{L}^2(\mathbb{R}^2)} \)
- 2D feature map: \( f_{in} \)
- 3D (SE(2)) feature map (after ReLU): \( f_{out} \)

Translation: \( k(R^{-1}_\theta (x' - x)) \)
Rotation: \( \theta \)
SE(2) equivariant cross-correlations

Lifting correlations: \((k \ast f)(x, \theta) = (\mathcal{L}^{SE(2) \rightarrow L_2(\mathbb{R}^2)}_g k \ast f)_{L_2(\mathbb{R}^2)} = (\mathcal{L}^{SO(2) \rightarrow L_2(\mathbb{R}^2)}_\theta k \ast f)_{L_2(\mathbb{R}^2)}\)

\[k(\mathbb{R}_\theta^{-1}(x' - x))\]

Representation of the roto-translation group!

Rotated 2D convolution kernel

\[\mathcal{L}^{SO(2) \rightarrow L_2(\mathbb{R}^2)}_\theta\]

2D feature map

\(f_{\text{in}}\)

3D (SE(2)) feature map (after ReLU)

\(f_{\text{out}}\)
Equivariance

SE(2) group lifting convolutions are roto-translation equivariant

$\mathcal{L}_{\theta}^{SO(2) \rightarrow \mathbb{L}_2(\mathbb{R}^2)} \Phi \rightarrow \mathcal{L}_{\theta}^{SO(2) \rightarrow \mathbb{L}_2(SE(2))}$

$(k \tilde{\star} f)(x, \theta) = (\mathcal{L}_{x}^{\mathbb{R}^2 \rightarrow \mathbb{L}_2(\mathbb{R}^2)} \mathcal{L}_{\theta}^{SO(2) \rightarrow \mathbb{L}_2(\mathbb{R}^2)} k, f)_{\mathbb{L}_2(\mathbb{R}^2)}$

planar rotation

periodic shift
Equivariance

SE(2) group lifting convolutions are roto-translation equivariant

\[ (k \star f)(x, \theta) = (\mathcal{L}_x^{\mathbb{R}^2 \rightarrow \mathbb{L}_2(\mathbb{R}^2)} \mathcal{L}_\theta^{SO(2) \rightarrow \mathbb{L}_2(\mathbb{R}^2)} k \cdot f)_{\mathbb{L}_2(\mathbb{R}^2)} \]

\[ \mathcal{L}_\theta^{SO(2) \rightarrow \mathbb{L}_2(\mathbb{SE}(2))} \]

planar rotation

periodic shift

planar rotation
Equivariance

SE(2) group lifting convolutions are roto-translation equivariant

\[ (k \star f)(x, \theta) = (\mathcal{L}_x^{R^2 \rightarrow L_2(R^2)} \mathcal{L}_\theta^{SO(2) \rightarrow L_2(R^2)} k \circ f)_{L_2(R^2)} \]

What about subsequent layers?
SE(2) equivariant cross-correlations

Group correlations:

\[(k \star f)(x, \theta) = \langle \mathcal{L}_{g}^{\text{SE}(2) \to \mathbb{L}^2(\text{SE}(2))} k, f \rangle_{\mathbb{L}^2(\text{SE}(2))} = \langle \mathcal{L}_{x}^{\mathbb{R}^2 \to \mathbb{L}^2(\text{SE}(2))} \mathcal{L}_{\theta}^{\text{SO}(2) \to \mathbb{L}^2(\text{SE}(2))} k, f \rangle_{\mathbb{L}^2(\text{SE}(2))} \]

translation  rotation

\(k(R_{\theta}^{-1}(x' - x), R_{\theta'} - \theta)\)
SE(2) equivariant cross-correlations

Group correlations:

\[(k \star f)(x, \theta) = (\mathcal{L}_g^{SE(2) \rightarrow \mathbb{L}_2(SE(2))} k , f)_{\mathbb{L}_2(SE(2))} = (\mathcal{L}_x^{\mathbb{R}^2 \rightarrow \mathbb{L}_2(SE(2))} \mathcal{L}_\theta^{SO(2) \rightarrow \mathbb{L}_2(SE(2))} k , f)_{\mathbb{L}_2(SE(2))}\]

Translation: \[k(R_{\theta}^{-1}(x' - x), R_{\theta' - \theta})\]
Rotation: \[\mathcal{L}_\theta^{SO(2) \rightarrow \mathbb{L}_2(SE(2))} k\]

Rotated SE(2) convolution kernel
SE(2) equivariant cross-correlations

\[(k \star f)(x, \theta) = (\mathcal{L}_g^{SE(2) \rightarrow \mathbb{L}_2(SE(2))} k, f)_{\mathbb{L}_2(SE(2))} = (\mathcal{L}_x^{R^2 \rightarrow \mathbb{L}_2(SE(2))} \mathcal{L}_\theta^{SO(2) \rightarrow \mathbb{L}_2(SE(2))} k, f)_{\mathbb{L}_2(SE(2))}\]

Group correlations:
- Planar rotation
- Periodic shift

Rotated SE(2) convolution kernel

**SE(2) feature map**

\[f_{in} \rightarrow f_{out} \text{ (after ReLU)}\]
SE(2) equivariant cross-correlations

\[(k \star f)(x, \theta) = (\mathcal{L}^{SE(2)\rightarrow\mathbb{L}_2(SE(2))}_g k \cdot f)_{\mathbb{L}_2(SE(2))} = (\mathcal{L}^{R^2\rightarrow\mathbb{L}_2(SE(2))}_x \mathcal{L}^{SO(2)\rightarrow\mathbb{L}_2(SE(2))}_\theta k \cdot f)_{\mathbb{L}_2(SE(2))}\]

Group correlations:

- **Translation**
  
  \[k(R^{-1}_\theta(x'-x), R_{\theta'}\theta)\]

- **Rotation**
  
  \[\mathcal{L}^{SO(2)\rightarrow\mathbb{L}_2(SE(2))}_\theta\]

Rotated SE(2) convolution kernel

SE(2) feature map

SE(2) feature map (after ReLU)
Equivariance

SE(2) group convolutions are roto-translation equivariant

\[
(k \ast f)(x, \theta) = (\mathcal{L}_x^{\mathbb{R}^2 \to \mathbb{L}_2(\text{SE}(2))} \mathcal{L}_\theta^{\text{SO}(2) \to \mathbb{L}_2(\text{SE}(2))} k, f)_{\mathbb{L}_2(\text{SE}(2))}
\]
Equivariance

SE(2) group convolutions are roto-translation equivariant

\( (k \ast f)(x, \theta) = (\mathcal{L}_x^{\mathbb{R}^2 \to \mathbb{L}_2(\text{SE}(2))} \mathcal{L}_\theta^{\text{SO}(2) \to \mathbb{L}_2(\text{SE}(2))} k \cdot f)_{\mathbb{L}_2(\text{SE}(2))} \)
2D cross-correlation (translation equivariant)

\[(k \star_{\mathbb{R}^2} f)(\mathbf{x}) = (\mathcal{L}_{\mathbf{x}}^{\mathbb{R}^2 \to \mathcal{L}^2(\mathbb{R}^2)}) k, f\}_{\mathbb{L}^2(\mathbb{R}^2)}
\]

\[= \int_{\mathbb{R}^2} k(\mathbf{x}' - \mathbf{x}) f(\mathbf{x}') d\mathbf{x}'\]
2D cross-correlation (translation equivariant)

\[
\begin{align*}
(k \star_{\mathbb{R}^2} f)(x) &= (\mathcal{L}^{\mathbb{R}^2 \rightarrow \mathbb{L}^2(\mathbb{R}^2)}_x k, f)_{\mathbb{L}^2(\mathbb{R}^2)} \\
&= \int_{\mathbb{R}^2} k(x' - x)f(x')\,dx'
\end{align*}
\]

SE(2) lifting correlations (roto-translation equivariant)

\[
(k \tilde{*} f)(x, \theta) = (\mathcal{L}^{SE(2) \rightarrow \mathbb{L}^2(\mathbb{R}^2)}_g k, f)_{\mathbb{L}^2(\mathbb{R}^2)} \\
= \int_{\mathbb{R}^2} k(R_\theta^{-1}(x' - x))f(x')\,dx'
\]
2D cross-correlation (translation equivariant)

\[(k \star_{\mathbb{R}^2} f')(x) = (\mathcal{L}_{x}^{\mathbb{R}^2 \rightarrow L_2(\mathbb{R}^2)} k, f)_{L_2(\mathbb{R}^2)} \]
\[= \int_{\mathbb{R}^2} k(x' - x)f(x')dx'\]

SE(2) lifting correlations (roto-translation equivariant)

\[(k \tilde{\star} f)(x, \theta) = (\mathcal{L}_{g}^{SE(2) \rightarrow L_2(SE(2))} k, f)_{L_2(SE(2))} \]
\[= \int_{\mathbb{R}^2} k(R_{\theta}^{-1}(x' - x))f(x')dx'\]

SE(2) G-correlations (roto-translation equivariant)

\[(k \tilde{\star} f)(x, \theta) = (\mathcal{L}_{g}^{SE(2) \rightarrow L_2(SE(2))} k, f)_{L_2(SE(2))} \]
\[= \int_{\mathbb{R}^2} \int_{S^1} k(R_{\theta}^{-1}(x' - x), \theta' - \theta \mod 2\pi)f(x', \theta')dx'd\theta'\]
Roto-translation group $SE(2) = \mathbb{R}^2 \times SO(2)$

2D feature map

Lifting layer

Group conv layer

Projection layer
Roto-translation group $SE(2) = \mathbb{R}^2 \times SO(2)$

2D feature map

Using a set of transformed 2D conv kernels

\[ \theta = \frac{\pi}{2} \]

\[ \theta = \frac{\pi}{4} \]

\[ \theta = 0 \]

G feature map (activation for oriented structures at each position and rotation)

G-feature maps are equivariant w.r.t. translation and rotation of the input

Lifting layer  \rightarrow  Group conv layer  \rightarrow  Projection layer
Roto-translation group $SE(2) = \mathbb{R}^2 \times SO(2)$

2D feature map

Using a set of transformed 2D conv kernels

$\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$

$\theta = 0$

Lifting layer

Group conv layer

Projection layer

G feature map (activation for oriented structures at each position and rotation)

G feature map (activation for faces at each position and rotation)

G-feature maps are equivariant w.r.t. translation and rotation of the input
Roto-translation group $SE(2) = \mathbb{R}^2 \times SO(2)$

2D feature map

Using a set of transformed 2D conv kernels

$\theta = \frac{\pi}{2}$
$\theta = \frac{\pi}{4}$
$\theta = 0$

Lifting layer

G feature map (activation for oriented structures at each position and rotation)

G-feature maps are equivariant w.r.t. translation and rotation of the input

$\theta = \frac{\pi}{4}$

$\theta = 0$

Group conv layer

Projection layer

Projection over sub-group $H$ guarantees local invariance

2D feature map
Roto-translation group $SE(2) = \mathbb{R}^2 \times SO(2)$

2D feature map

Using a set of transformed 2D conv kernels

$\theta = \frac{\pi}{2}$
$\theta = \frac{\pi}{4}$
$\theta = 0$

Lifting layer

G feature map (activation for oriented structures at each position and rotation)

G-feature maps are equivariant w.r.t. translation and rotation of the input

$\theta = \frac{\pi}{4}$
$\theta = 0$

Group conv layer

Projection over sub-group $H$ guarantees local invariance

Projection layer

Scale-translation group $\mathbb{R}^2 \times \mathbb{R}^+$
Roto-translation group $SE(2) = \mathbb{R}^2 \times SO(2)$

2D feature map

Using a set of transformed 2D conv kernels

$\theta = \frac{\pi}{2}$
$\theta = \frac{\pi}{4}$
$\theta = 0$

G feature map (activation for oriented structures at each position and rotation)

G-feature maps are equivariant w.r.t. translation and rotation of the input

Group conv layer

C-feature maps are equivariant w.r.t. translation and scaling of the input

Projection layer

Projection over sub-group $H$ guarantees local invariance

Scale-translation group $\mathbb{R}^2 \times \mathbb{R}^+$
Roto-translation group $SE(2) = \mathbb{R}^2 \times SO(2)$

2D feature map

Using a set of transformed 2D conv kernels

\[ \theta = \frac{\pi}{2} \]
\[ \theta = \frac{\pi}{4} \]
\[ \theta = 0 \]

Lifting layer

Scale-translation group $\mathbb{R}^2 \times \mathbb{R}^+$

\[ s = 2 \]
\[ s = 1.4 \]
\[ s = 1 \]

G-feature map (activation for oriented structures at each position and rotation)

\[ \text{G-feature maps are equivariant w.r.t. translation and rotation of the input} \]

Group conv layer

Activation for circles at each position and scale

G-feature map (activation for faces at each position and rotation)

\[ \theta = \frac{\pi}{4} \]
\[ \theta = 0 \]

Using a set of transformed G-conv kernels

Projection over sub-group $H$ guarantees local invariance

Projection layer

Activation for faces at each position and scale

C-conv kernels assign weights to activations in a pattern of relative poses
Roto-translation group $SE(2) = \mathbb{R}^2 \times SO(2)$

2D feature map

Using a set of transformed 2D conv kernels

$\theta = \frac{\pi}{2}$
$\theta = \frac{\pi}{4}$
$\theta = 0$

Lifting layer

G-feature maps (activation for oriented structures at each position and rotation)

G-feature maps are equivariant w.r.t. translation and rotation of the input

Group conv layer

$\theta = \frac{\pi}{4}$
$\theta = 0$

C-feature maps are equivariant w.r.t. translation and scaling of the input

$\text{scale}$

$G$-conv kernels assign weights to activations in a pattern of relative poses

Activation for circles at each position and scale

Projection layer

Projection over sub-group $H$ guarantees local invariance

Activation for faces at each position and scale

2D feature map
Summary

• Group convolutional neural networks intuitively perform template matching.

• A template (kernel) is transformed and matched (inner-product) under all possible transformations in the group.

• This creates higher-dimensional feature maps (functions on the group) on which we again define template matching via the group action.

• In these higher dimensional feature maps we can detect advanced patterns in terms of features at relative poses!

• G-CNNs are based on equivariant layers (thus weight sharing) and guarantee invariance through pooling.