Group Equivariant Deep Learning

Lecture 1 - Regular group convolutions

Lecture 1.2 - Group Theory | The basics

Preliminaries for regular group convolutions

Groups, group product, inverse, action, representation, affine groups $G = \mathbb{R}^d \rtimes H$

Erik Bekkers, Amsterdam Machine Learning Lab, University of Amsterdam

This mini-course serves as a module with the UvA Master AI course Deep Learning 2 https://uvadl2c.github.io/

What is a group?

A group $(G, \cdot\,)$ is a set of elements G equipped with a group product $\cdot$, a binary operator, that satisfies the following four axioms:

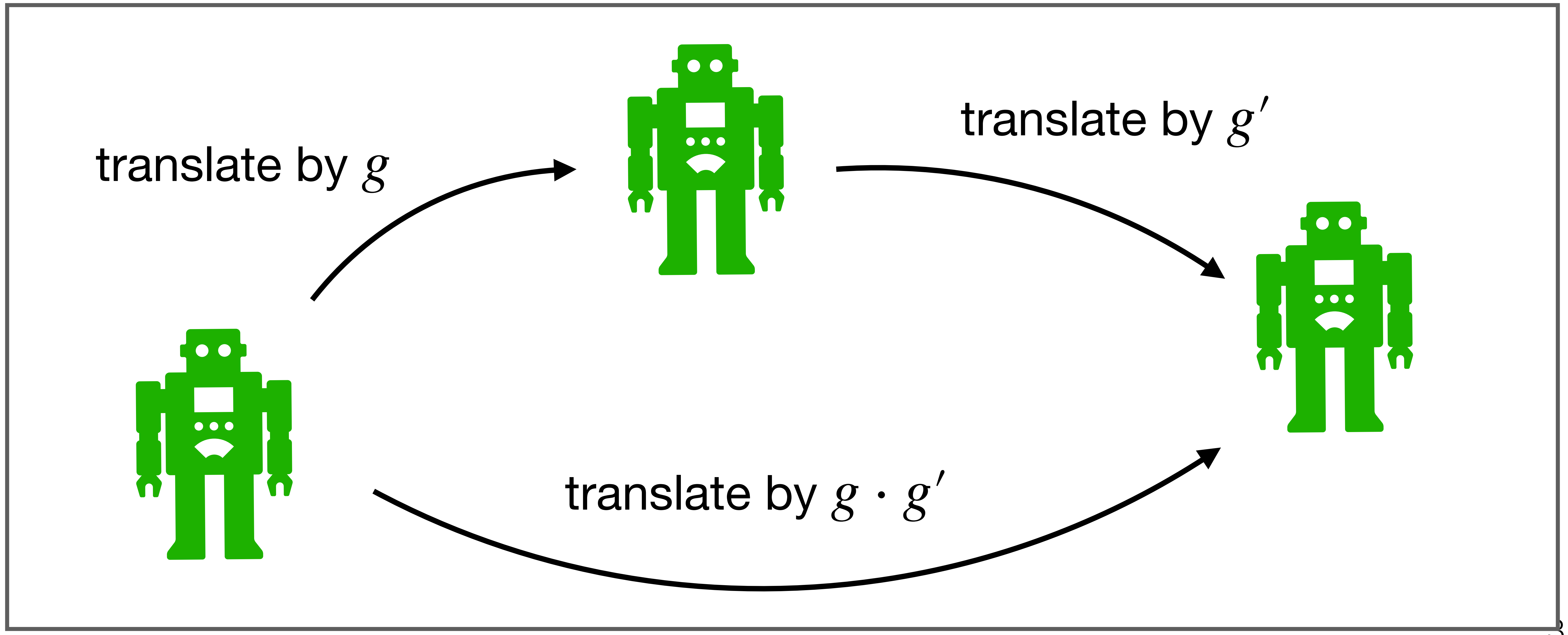
- **Closure**: Given two elements g and h of G, the product $g \cdot h$ is also in G.
- **Associativity**: For $g, h, i \in G$ the product $\cdot$ is associative, i.e., $g \cdot (h \cdot i) = (g \cdot h) \cdot i$.
- **Identity element**: There exists an identity element $e \in G$ such that $e \cdot g = g \cdot e = g$ for any $g \in G$.
- **Inverse element**: For each $g \in G$ there exists an inverse element $g^{-1} \in G$ s.t. $g^{-1} \cdot g = g \cdot g^{-1} = e$.

Translation group $(\mathbb{R}^2, +)$

The translation group consists of all possible translations in $\mathbb{R}^2$ and is equipped with the group product and group inverse:

$$g \cdot g' = (\mathbf{x} + \mathbf{x}')$$
$$g^{-1} = (-\mathbf{x})$$

with $g = (\mathbf{x}), g' = (\mathbf{x}')$ and $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^2$.



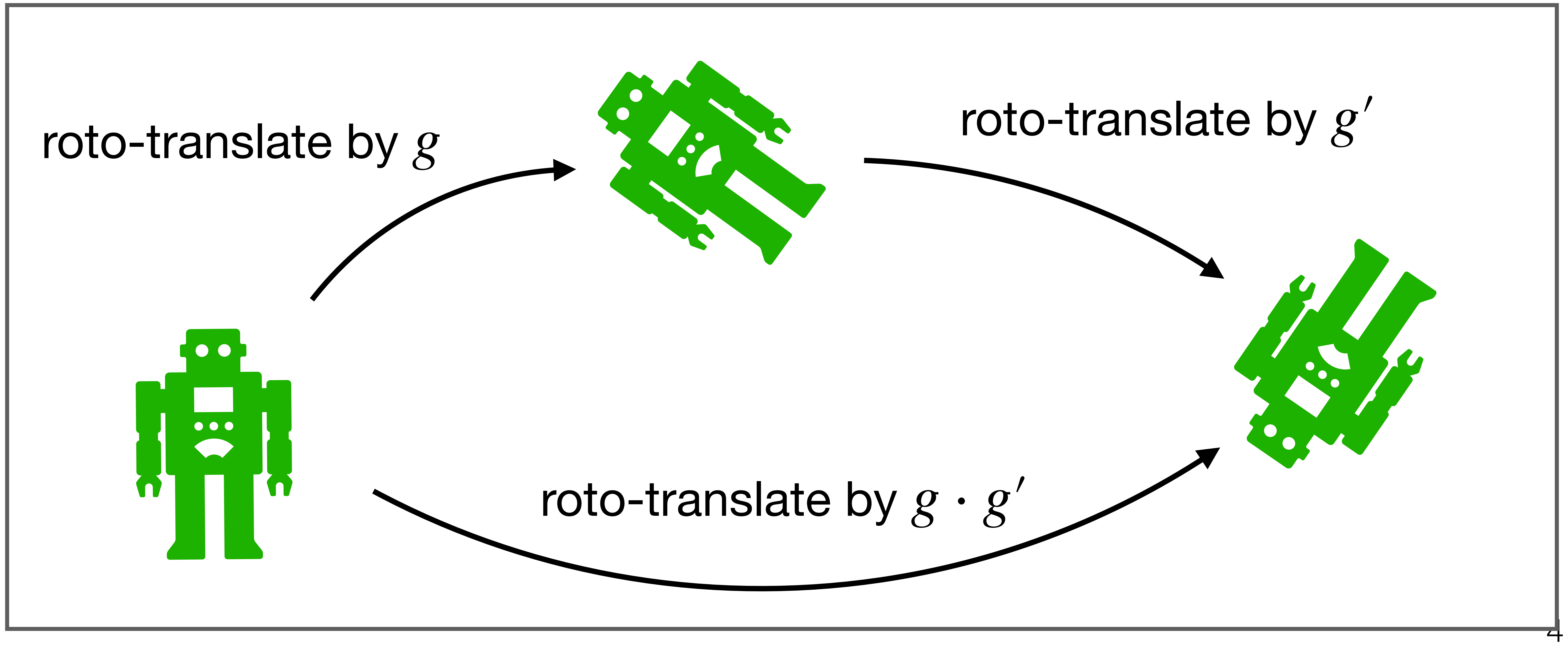
Roto-ranslation group $SE(2)$

*2D **S**pecial **E**uclidean motion group*

The group $SE(2) = \mathbb{R}^2 \rtimes SO(2)$ consists of the coupled space $\mathbb{R}^2 \times S^1$ of translations vectors in $\mathbb{R}^2$, and rotations in $SO(2)$ (or equivalently orientations in S^1), and is equipped with the group product and group inverse:

$$g \cdot g' = (\mathbf{x}, \mathbf{R}_\theta) \cdot (\mathbf{x}', \mathbf{R}_{\theta'}) = (\mathbf{R}_\theta \mathbf{x}' + \mathbf{x}, \mathbf{R}_{\theta+\theta'})$$
$$g^{-1} = (-\mathbf{R}_\theta^{-1}\mathbf{x}, \mathbf{R}_\theta^{-1})$$

with $g = (\mathbf{x}, \mathbf{R}_\theta)$, $g' = (\mathbf{x}', \mathbf{R}_{\theta'})$.



Roto-ranslation group $SE(2)$

*2D **S**pecial **E**uclidean motion group*

Matrix representation: The group can also be represented by matrices

$$g = (\mathbf{x}, \mathbf{R}_\theta) \quad \leftrightarrow \quad \mathbf{G} = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R}_\theta & \mathbf{x} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

with the group product and inverse simply given by the matrix product and matrix inverse.

In parametric form: $$(\mathbf{x}, \theta) \cdot (\mathbf{x}', \theta') = (\mathbf{R}_\theta \mathbf{x}' + \mathbf{x},\ \theta + \theta' \bmod 2\pi)$$

$$\leftrightarrow$$

In matrix form: $$\begin{pmatrix} \mathbf{R}_\theta & \mathbf{x} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}'_\theta & \mathbf{x}' \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R}_{\theta+\theta'} & \mathbf{R}_\theta \mathbf{x}' + \mathbf{x} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

Scale-translation group $R^2 \rtimes \mathbb{R}^+$

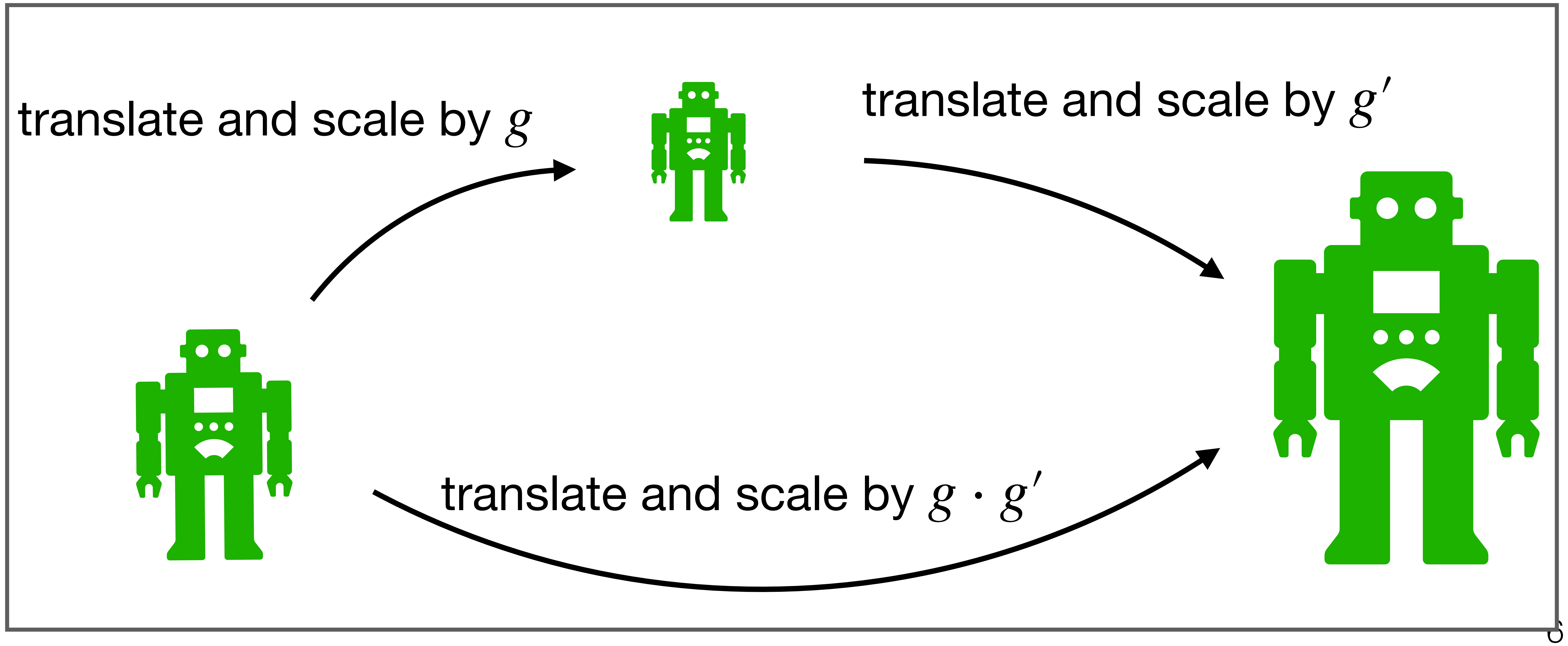
The scale-translation group of space $\mathbb{R}^2 \times \mathbb{R}^+$ of translations vectors in $\mathbb{R}^2$ and scale/dilation factors in $\mathbb{R}^+$, and is equipped with the group product and group inverse:

$$g \cdot g' = (\mathbf{x}, s) \cdot (\mathbf{x}', s') = (s\mathbf{x}' + \mathbf{x}, ss')$$
$$g^{-1} = \left(-\frac{1}{s}\mathbf{x}, \frac{1}{s}\right)$$

with $g = (\mathbf{x}, s), g' = (\mathbf{x}', s')$.

with $g \cdot g^{-1} = e = (\mathbf{0},1)$

matrix repr: $\mathbf{G} = \begin{pmatrix} \mathbf{I}_s & \mathbf{x} \\ \mathbf{0}^T & 1 \end{pmatrix}$



Affine groups $G = \mathbb{R}^d \rtimes H$

Affine groups are semi-direct product groups of some group H with an action on $\mathbb{R}^d$ from which we derive the following group product and inverse

transform/shift sub-group

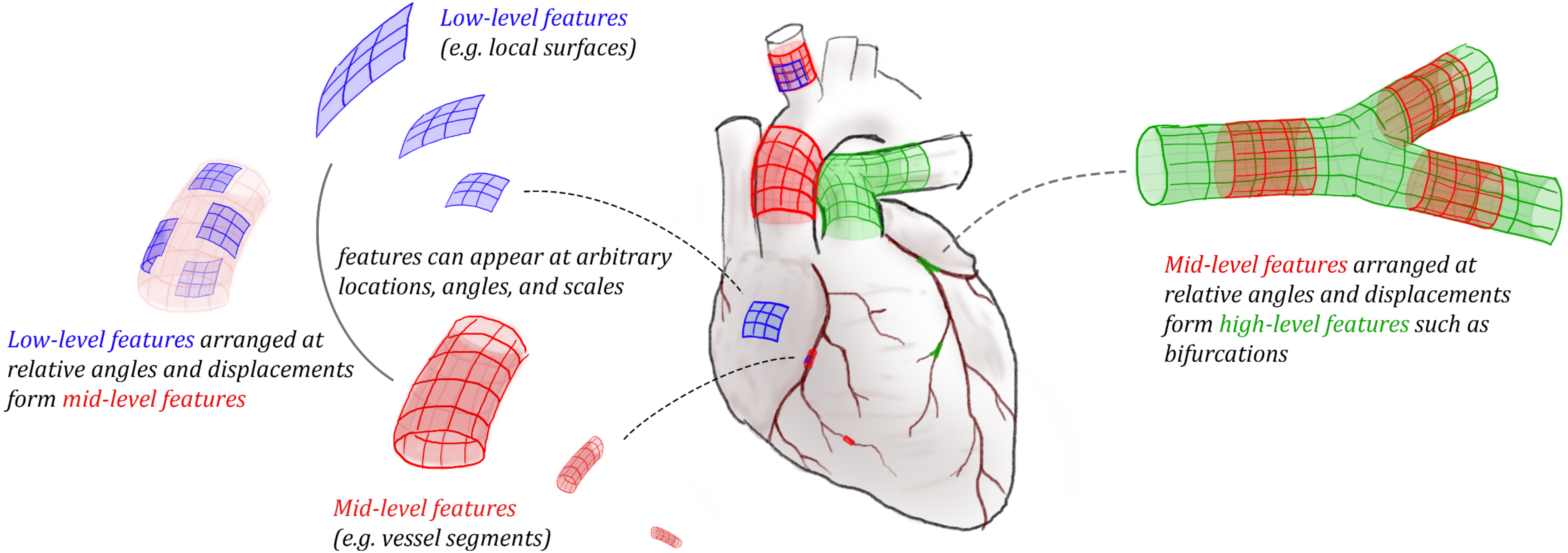
$$g \cdot g' = (\mathbf{x}, h) \cdot (\mathbf{x}', h') = (h \cdot \mathbf{x}' + \mathbf{x}, h \cdot h')$$

Transform point + shift point

$$g^{-1} = \left(-h^{-1} \cdot \mathbf{x}, h^{-1}\right)$$

with group elements $g = (\mathbf{x}, h), g' = (\mathbf{x}', h')$.

Psychology of vision: recognition by components

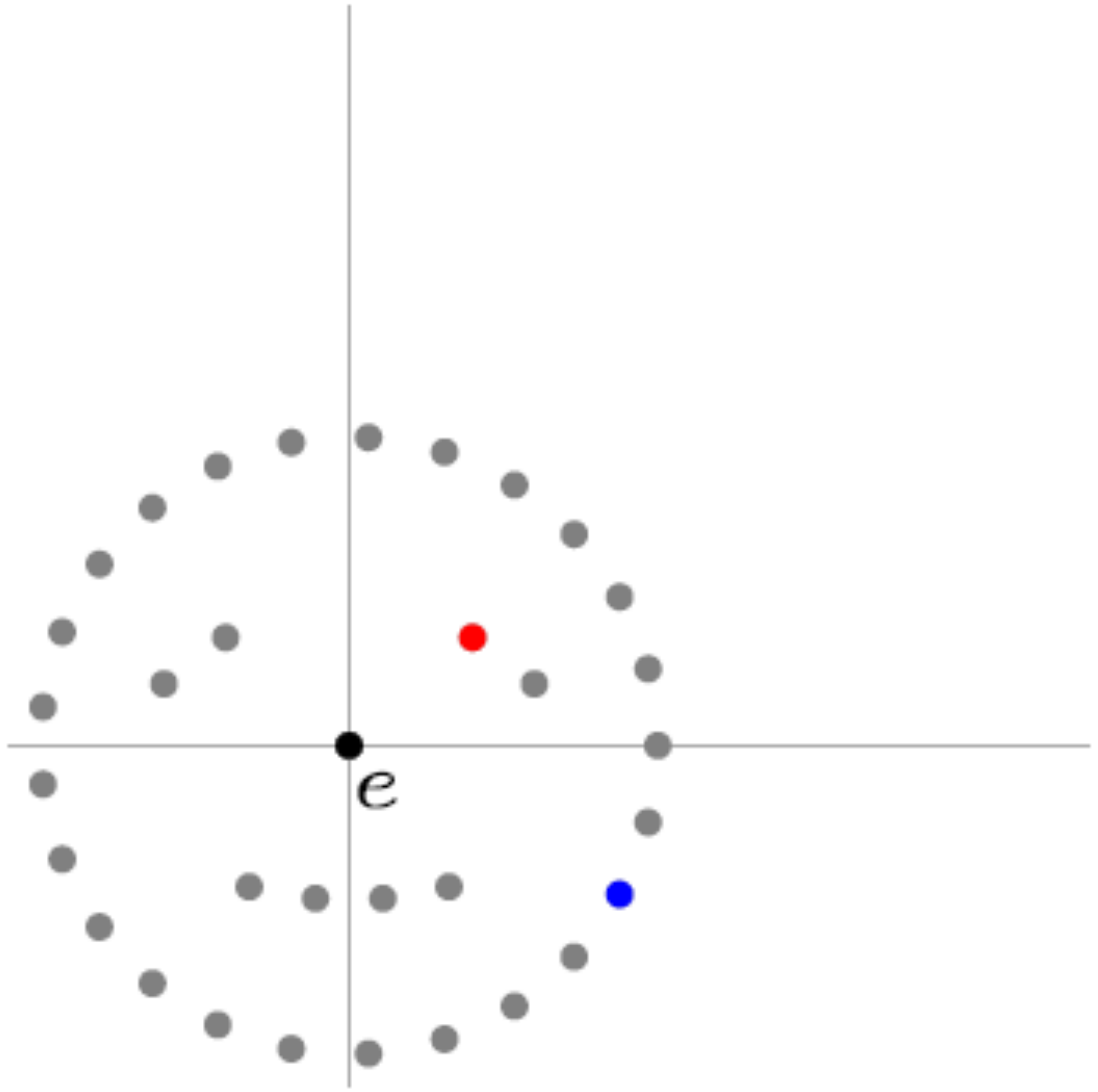


Translation group $(\mathbb{R}^2, +)$

The translation group consists of all possible translations in $\mathbb{R}^2$ and is equipped with the group product and group inverse:

$$g \cdot g' = (\mathbf{x} + \mathbf{x}')$$
$$g^{-1} = (-\mathbf{x})$$

with $g = (\mathbf{x}), g' = (\mathbf{x}')$ and $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^2$.

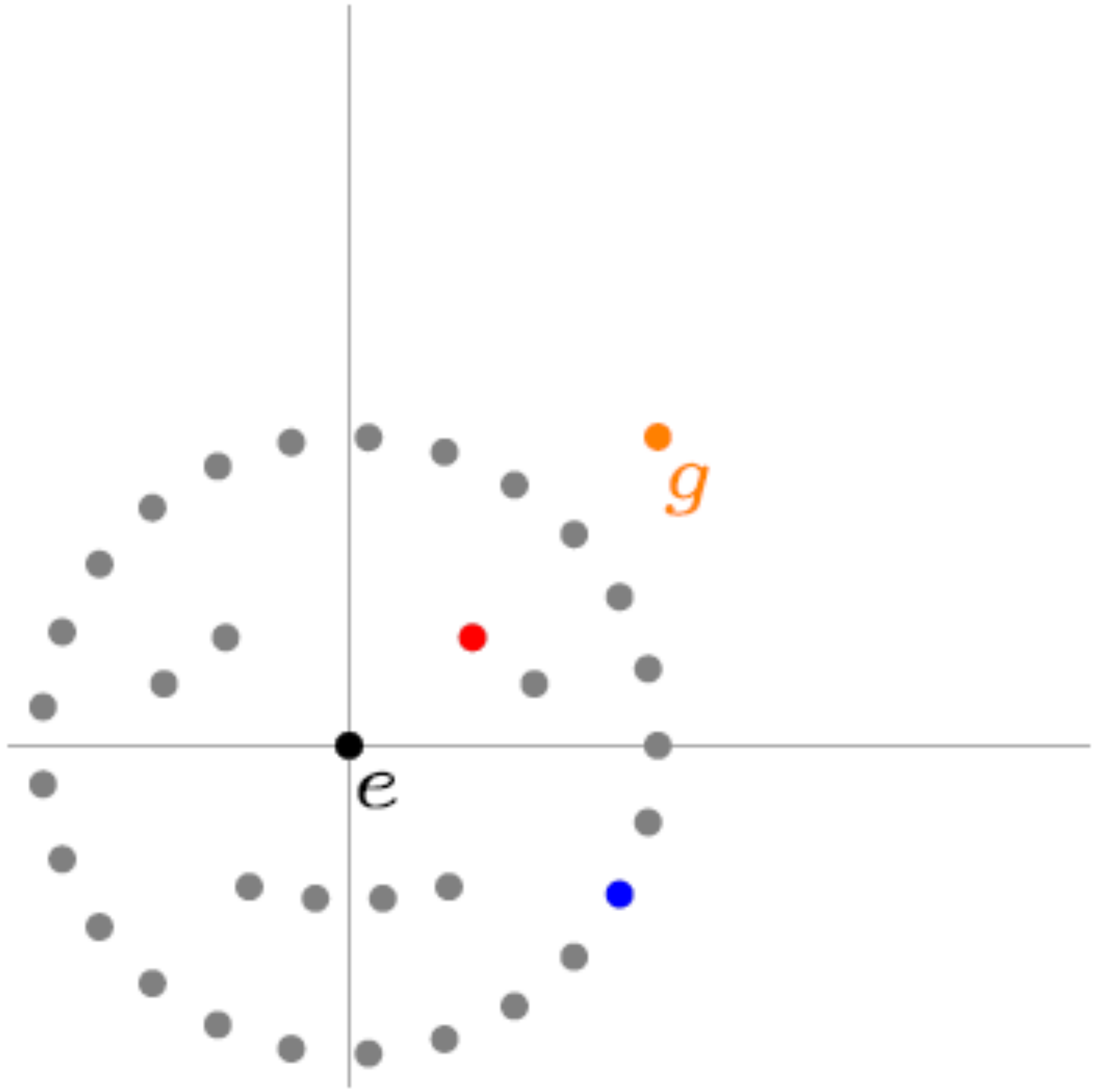


Translation group $(\mathbb{R}^2, +)$

The translation group consists of all possible translations in $\mathbb{R}^2$ and is equipped with the group product and group inverse:

$$g \cdot g' = (\mathbf{x} + \mathbf{x}')$$
$$g^{-1} = (-\mathbf{x})$$

with $g = (\mathbf{x}), g' = (\mathbf{x}')$ and $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^2$.

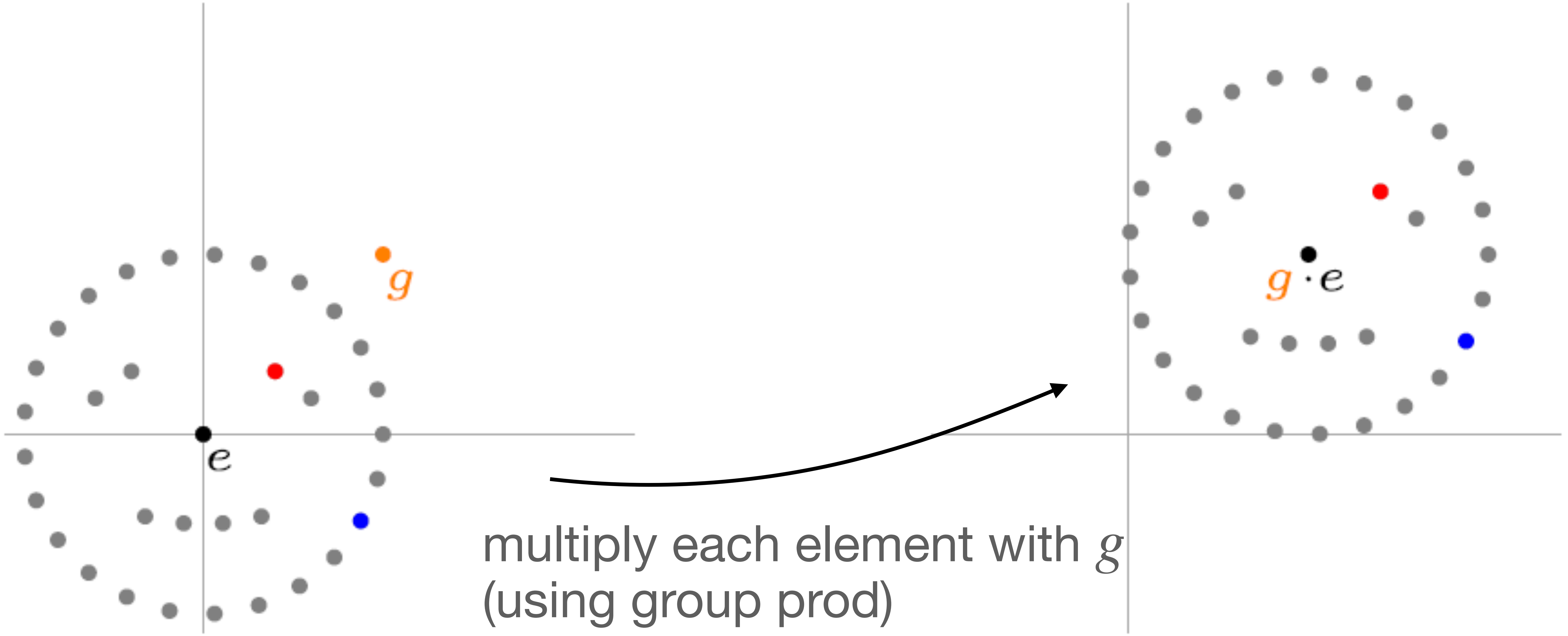


Translation group $(\mathbb{R}^2, +)$

The translation group consists of all possible translations in $\mathbb{R}^2$ and is equipped with the group product and group inverse:

$$g \cdot g' = (\mathbf{x} + \mathbf{x}')$$
$$g^{-1} = (-\mathbf{x})$$

with $g = (\mathbf{x}), g' = (\mathbf{x}')$ and $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^2$.



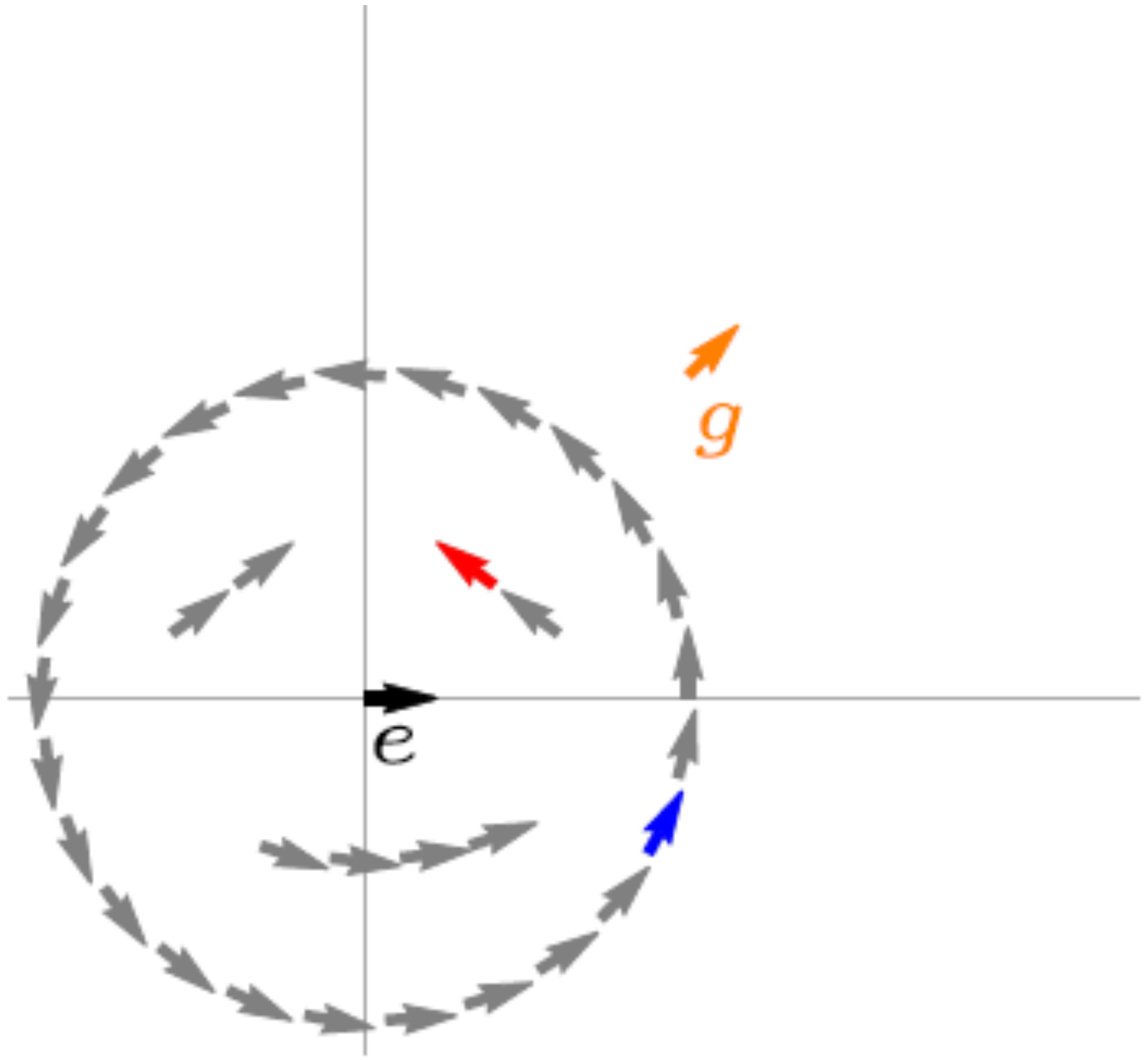
Roto-ranslation group $SE(2)$

*2D **S**pecial **E**uclidean motion group*

The group $SE(2) = \mathbb{R}^2 \rtimes SO(2)$ consists of the coupled space $\mathbb{R}^2 \times S^1$ of translations vectors in $\mathbb{R}^2$, and rotations in $SO(2)$ (or equivalently orientations in S^1), and is equipped with the group product and group inverse:

$$g \cdot g' = (\mathbf{x}, \mathbf{R}_\theta) \cdot (\mathbf{x}', \mathbf{R}_{\theta'}) = (\mathbf{R}_\theta \mathbf{x}' + \mathbf{x}, \mathbf{R}_{\theta+\theta'})$$
$$g^{-1} = (-\mathbf{R}_\theta^{-1}\mathbf{x}, \mathbf{R}_\theta^{-1})$$

with $g = (\mathbf{x}, \mathbf{R}_\theta)$, $g' = (\mathbf{x}', \mathbf{R}_{\theta'})$.



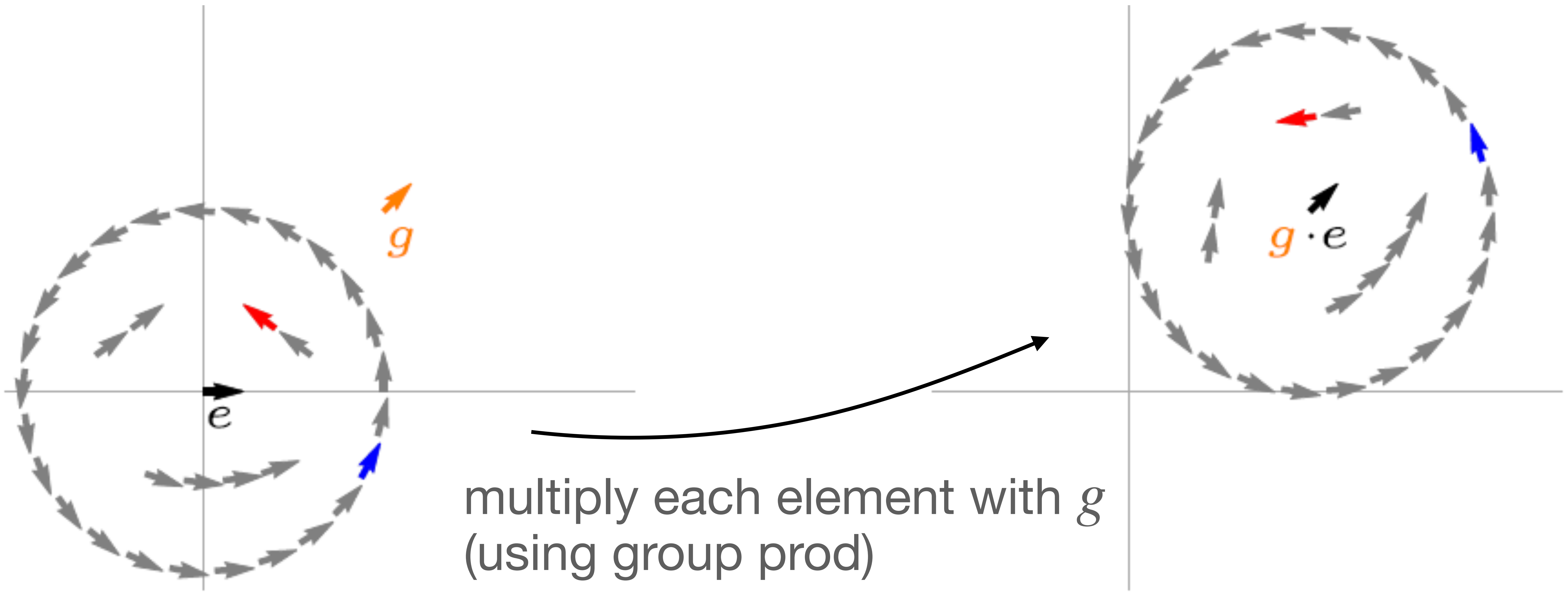
Roto-ranslation group $SE(2)$

2D **S**pecial **E**uclidean motion group

The group $SE(2) = \mathbb{R}^2 \rtimes SO(2)$ consists of the coupled space $\mathbb{R}^2 \times S^1$ of translations vectors in $\mathbb{R}^2$, and rotations in $SO(2)$ (or equivalently orientations in S^1), and is equipped with the group product and group inverse:

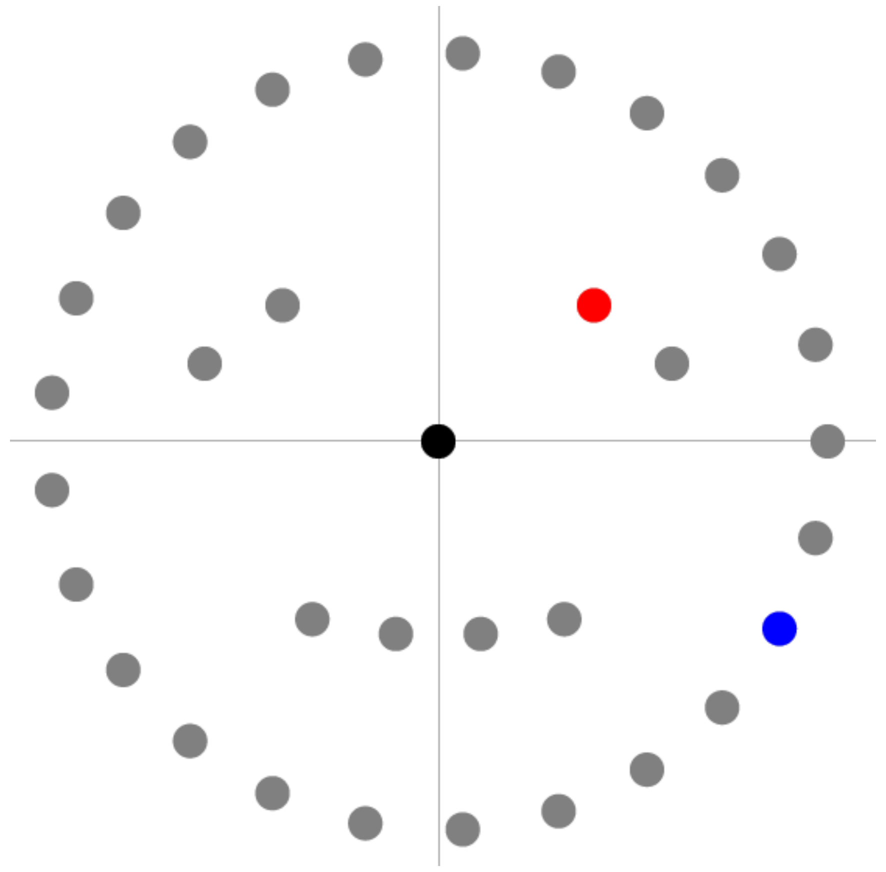
$$g \cdot g' = (\mathbf{x}, \mathbf{R}_\theta) \cdot (\mathbf{x}', \mathbf{R}_{\theta'}) = (\mathbf{R}_\theta \mathbf{x}' + \mathbf{x}, \mathbf{R}_{\theta+\theta'})$$
$$g^{-1} = (-\mathbf{R}_\theta^{-1}\mathbf{x}, \mathbf{R}_\theta^{-1})$$

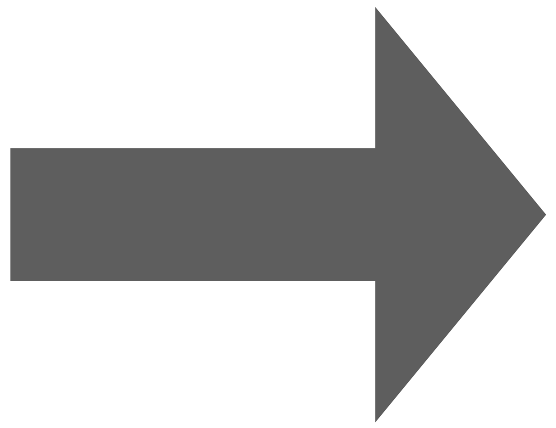
with $g = (\mathbf{x}, \mathbf{R}_\theta)$, $g' = (\mathbf{x}', \mathbf{R}_{\theta'})$.



So… How to translate this to (G-)CNNs?

Set of points (group elements)

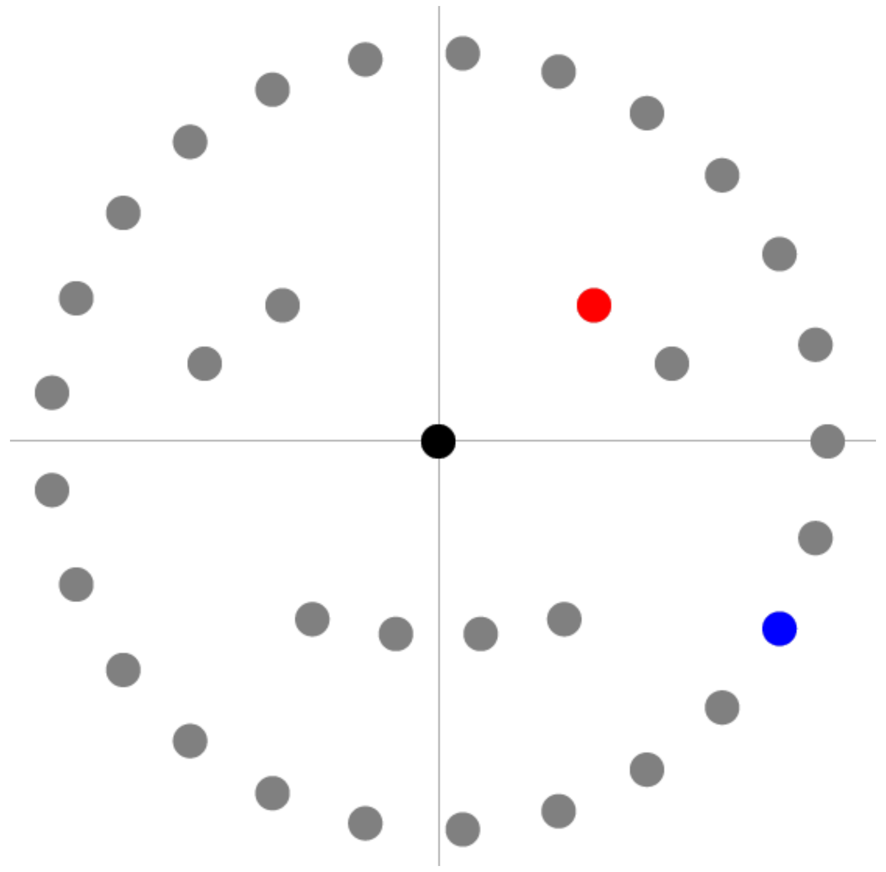




Convolution kernel

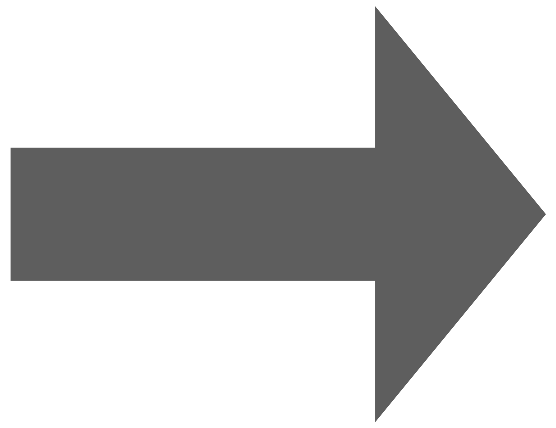
So… How to translate this to (G-)CNNs?

Set of points (group elements)



$\{g_1, g_2, \ldots\} \subset G = (\mathbb{R}^2, +)$

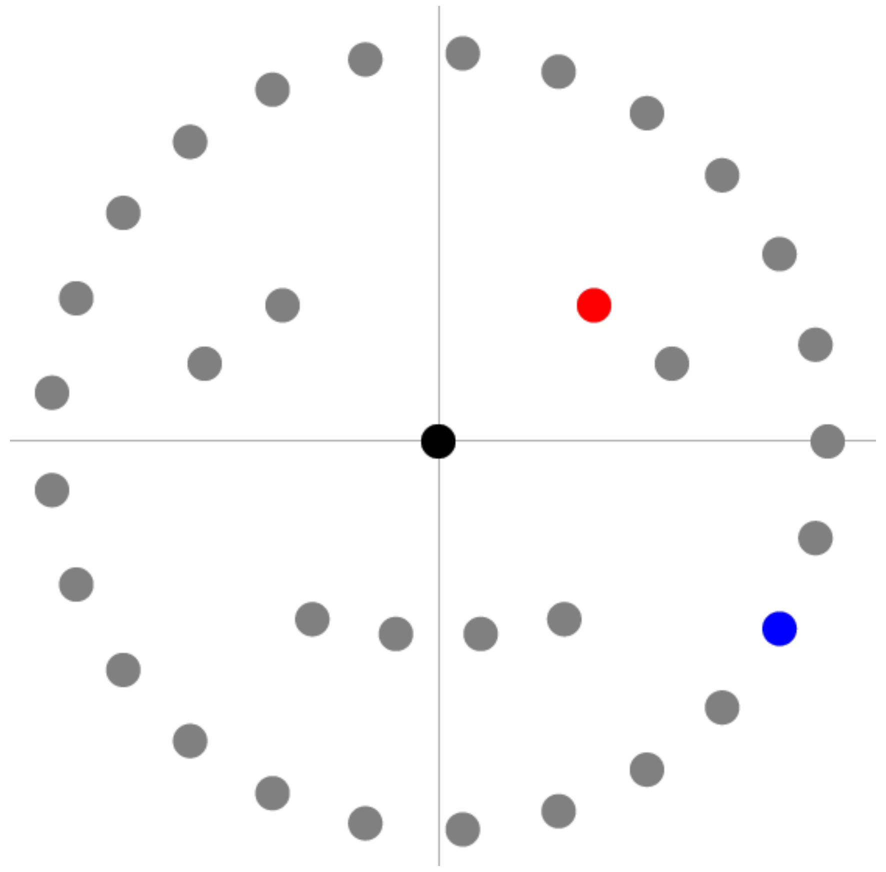
"A collection of parts in certain poses"



Convolution kernel

So… How to translate this to (G-)CNNs?

Set of points (group elements)



$\{g_1, g_2, \ldots\} \subset G = (\mathbb{R}^2, +)$

"A collection of parts in certain poses"

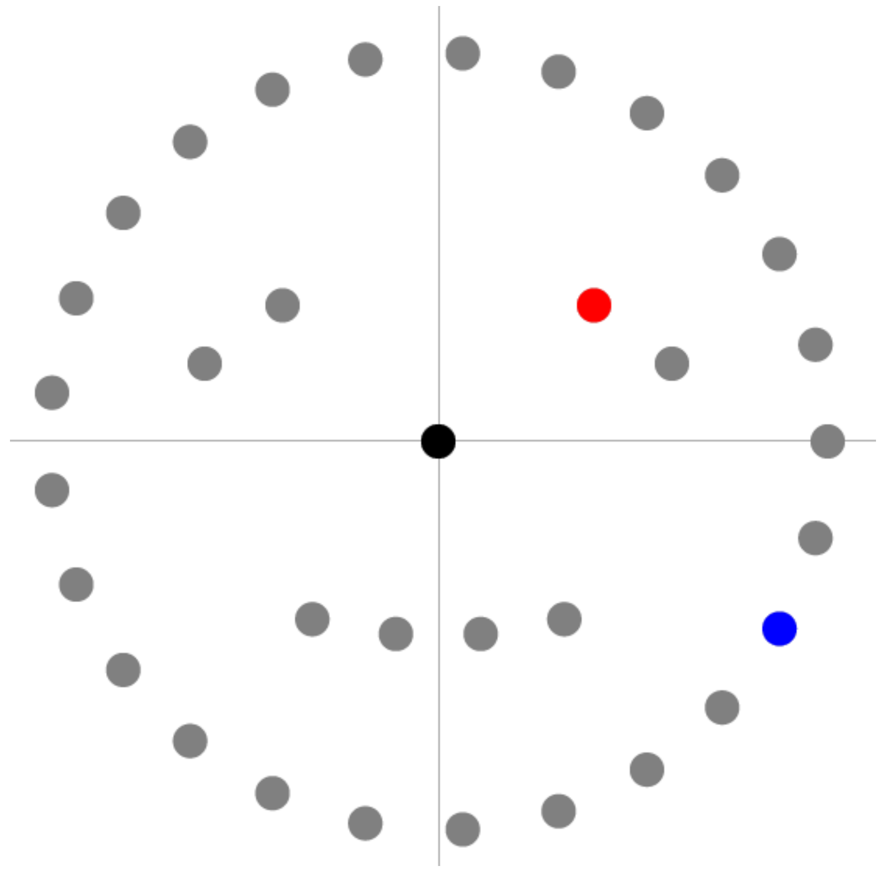
Convolution kernel

$k \in \mathbb{L}_2(\mathbb{R}^2)$

"Assigning weights to relative poses"

So… How to translate this to (G-)CNNs?

Set of points (group elements)



$$\{g_1, g_2, \ldots\} \subset G = (\mathbb{R}^2, +)$$

"A collection of parts in certain poses"

Transforms via group product

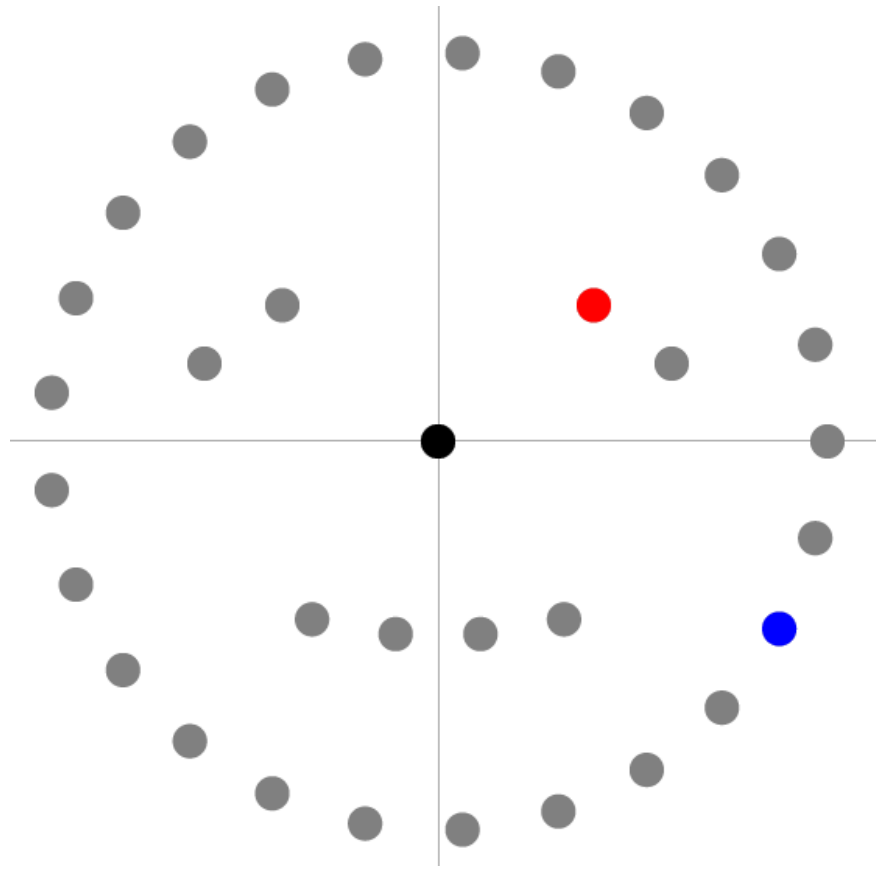
Convolution kernel

$$k \in \mathbb{L}_2(\mathbb{R}^2)$$

"Assigning weights to relative poses"

So… How to translate this to (G-)CNNs?

Set of points (group elements)



$\{g_1, g_2, \ldots\} \subset G = (\mathbb{R}^2, +)$

“A collection of parts in certain poses”

Transforms via group product

Convolution kernel

$k \in \mathbb{L}_2(\mathbb{R}^2)$

“Assigning weights to relative poses”

Transforms via group representations

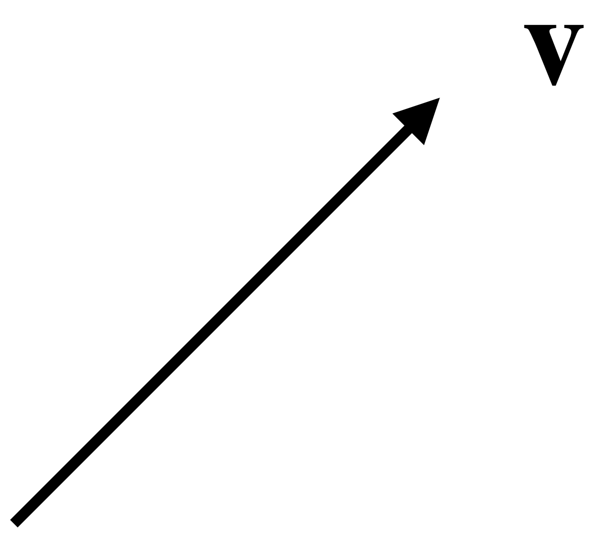
Representations

A representation $\rho : G \rightarrow GL(V)$ is a group homomorphism from G to the general linear group $GL(V)$.

That is $\rho(g)$ is a linear transformation that is parameterized by group elements $g \in G$ that transforms some vector $\mathbf{v} \in V$ (e.g. an image) such that

$$\rho(g') \circ \rho(g)[\mathbf{v}] = \rho(g' \cdot g)\,[\mathbf{v}]$$

Representations

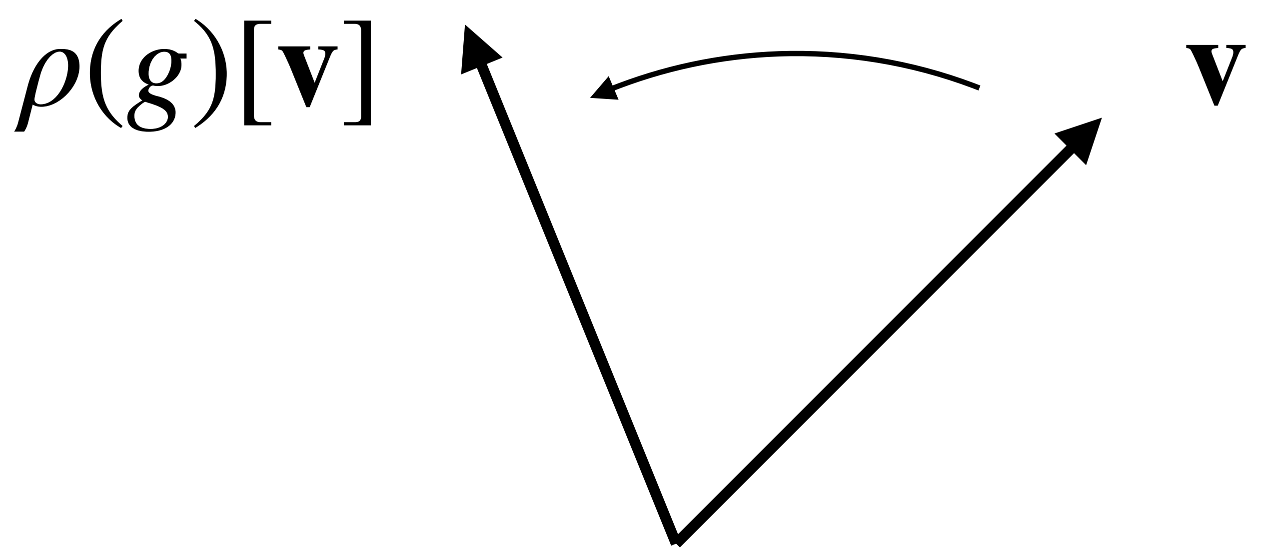


A representation $\rho : G \rightarrow GL(V)$ is a group homomorphism from G to the general linear group $GL(V)$.

That is $\rho(g)$ is a linear transformation that is parameterized by group elements $g \in G$ that transforms some vector $\mathbf{v} \in V$ (e.g. an image) such that

$$\rho(g') \circ \rho(g)[\mathbf{v}] = \rho(g' \cdot g)\,[\mathbf{v}]$$

Representations

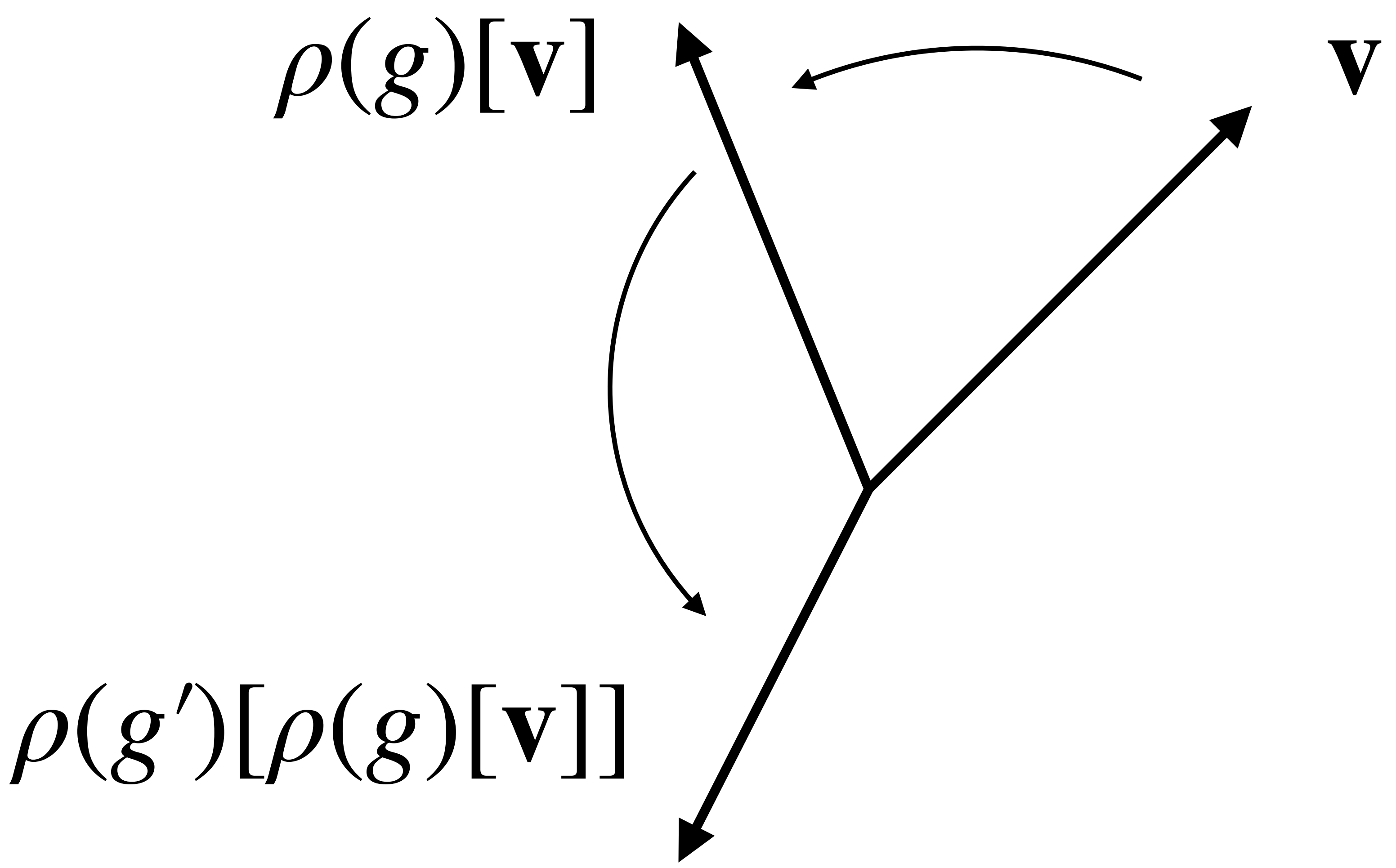


A representation $\rho : G \rightarrow GL(V)$ is a group homomorphism from G to the general linear group $GL(V)$.

That is $\rho(g)$ is a linear transformation that is parameterized by group elements $g \in G$ that transforms some vector $\mathbf{v} \in V$ (e.g. an image) such that

$$\rho(g') \circ \rho(g)[\mathbf{v}] = \rho(g' \cdot g)\,[\mathbf{v}]$$

Representations

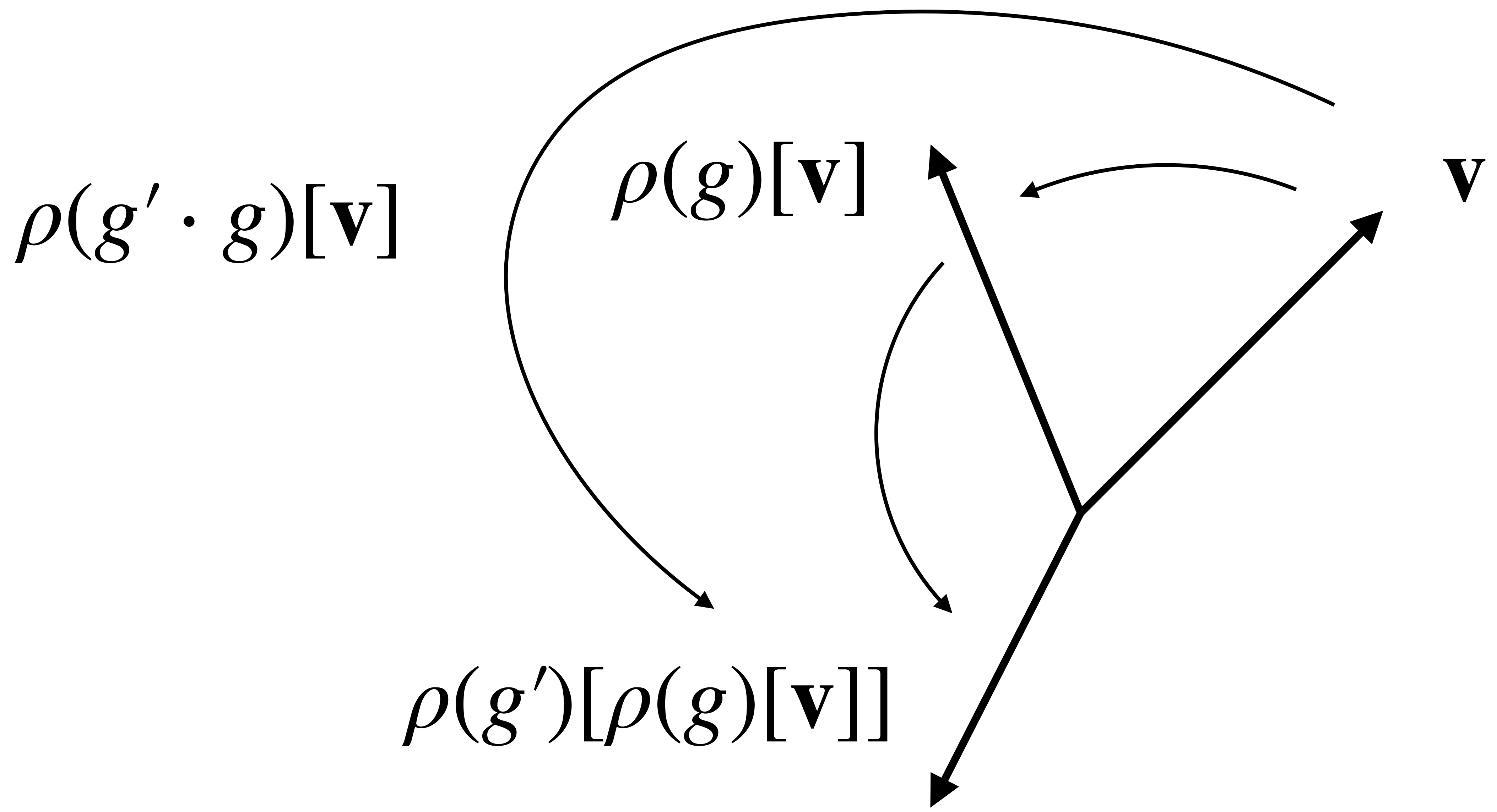


A representation $\rho : G \to GL(V)$ is a group homomorphism from G to the general linear group $GL(V)$.

That is $\rho(g)$ is a linear transformation that is parameterized by group elements $g \in G$ that transforms some vector $\mathbf{v} \in V$ (e.g. an image) such that

$$\rho(g') \circ \rho(g)[\mathbf{v}] = \rho(g' \cdot g)\,[\mathbf{v}]$$

Representations



A representation $\rho : G \to GL(V)$ is a group homomorphism from G to the general linear group $GL(V)$.

That is $\rho(g)$ is a linear transformation that is parameterized by group elements $g \in G$ that transforms some vector $\mathbf{v} \in V$ (e.g. an image) such that

$$\rho(g') \circ \rho(g)[\mathbf{v}] = \rho(g' \cdot g)\,[\mathbf{v}]$$

Left-regular Representations

A left-regular representation $\mathscr{L}_g$ is a representation that transforms functions f by transforming their domains via the inverse group action

$$\mathscr{L}_g[f](x) := f(g^{-1} \cdot x)$$

"group action" equals group product when domain is G

Left-regular Representations

Example:

$f \in \mathbb{L}_2(\mathbb{R}^2)$

- a 2D image

$G = SE(2)$

- the roto-translation group

$\mathscr{L}_g(f)(\mathbf{y}) = f(\mathbf{R}_\theta^{-1}(\mathbf{y} - \mathbf{x}))$

- a roto-translation of the image

A left-regular representation $\mathscr{L}_g$ is a representation that transforms functions f by transforming their domains via the inverse group action

$$\mathscr{L}_g[f](x) := f(g^{-1} \cdot x)$$

"group action" equals group product when domain is G

Left-regular Representations

Example:

$f \in \mathbb{L}_2(\mathbb{R}^2)$

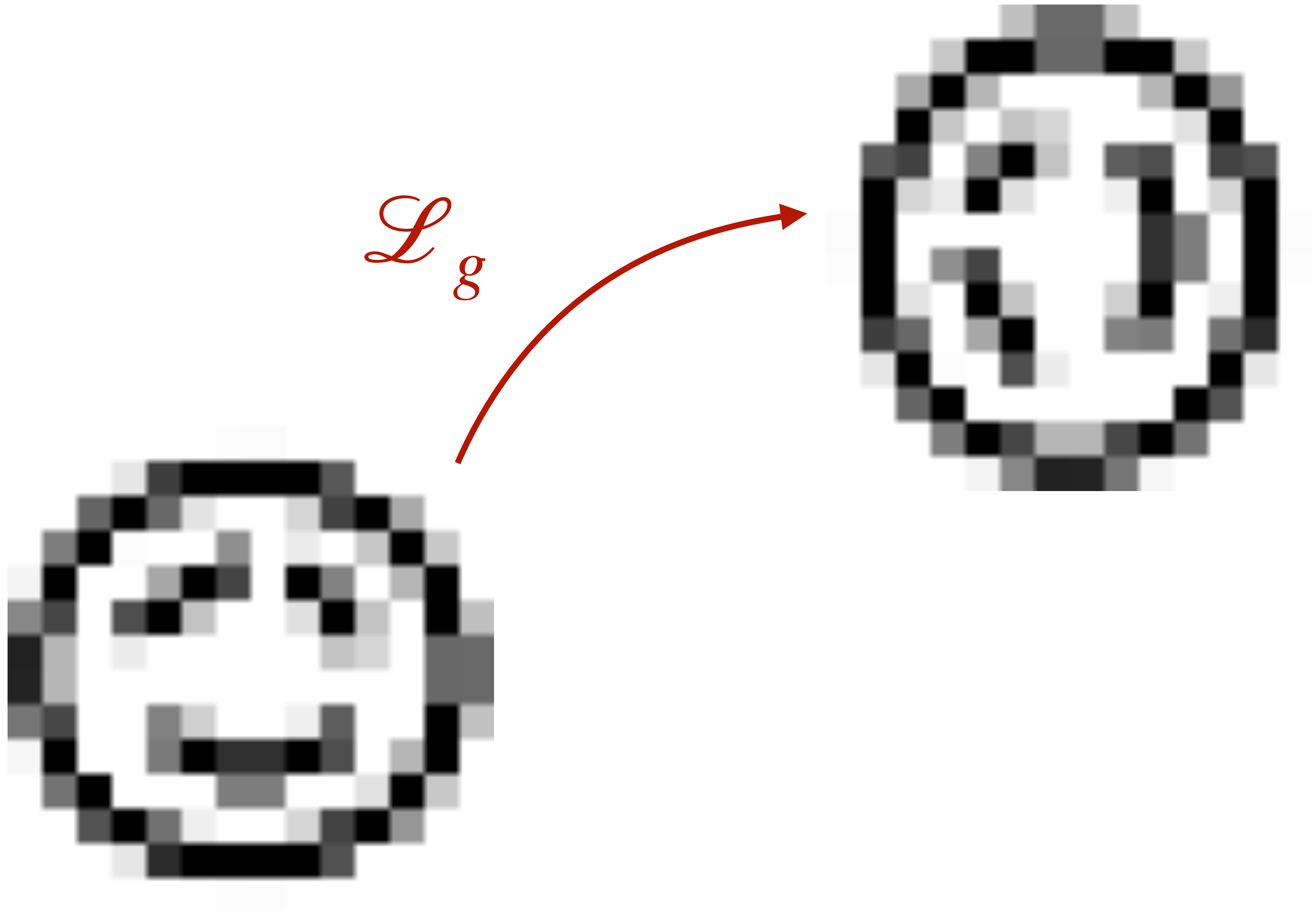
- a 2D image

$G = SE(2)$

- the roto-translation group

$\mathscr{L}_g(f)(\mathbf{y}) = f(\mathbf{R}_\theta^{-1}(\mathbf{y} - \mathbf{x}))$

- a roto-translation of the image



A left-regular representation $\mathscr{L}_g$ is a representation that transforms functions f by transforming their domains via the inverse group action

$$\mathscr{L}_g[f](x) := f(g^{-1} \cdot x)$$

"group action" equals group product when domain is G

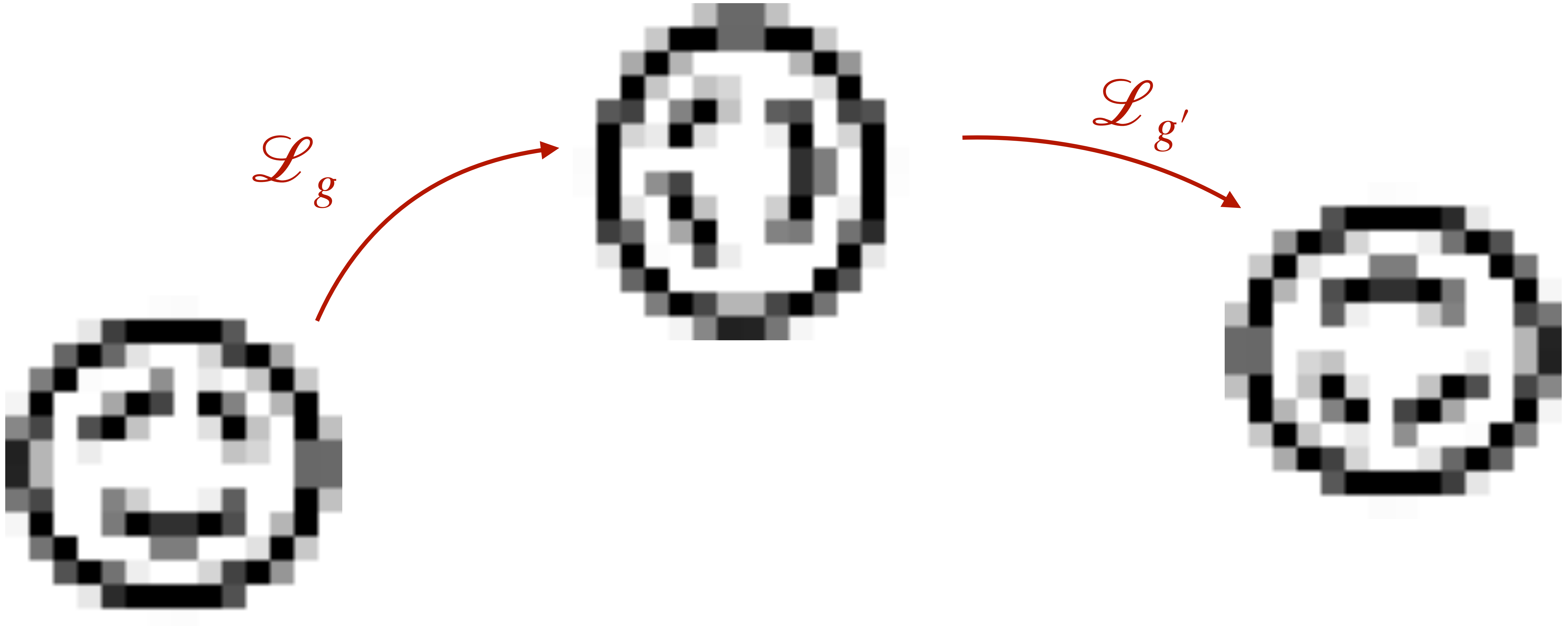
Left-regular Representations

Example:

$f \in \mathbb{L}_2(\mathbb{R}^2)$
- a 2D image

$G = SE(2)$
- the roto-translation group

$\mathscr{L}_g(f)(\mathbf{y}) = f(\mathbf{R}_\theta^{-1}(\mathbf{y} - \mathbf{x}))$
- a roto-translation of the image



A left-regular representation $\mathscr{L}_g$ is a representation that transforms functions f by transforming their domains via the inverse group action

$$\mathscr{L}_g[f](x) := f(g^{-1} \cdot x)$$

"group action" equals group product when domain is G

Left-regular Representations

Example:

$f \in \mathbb{L}_2(\mathbb{R}^2)$

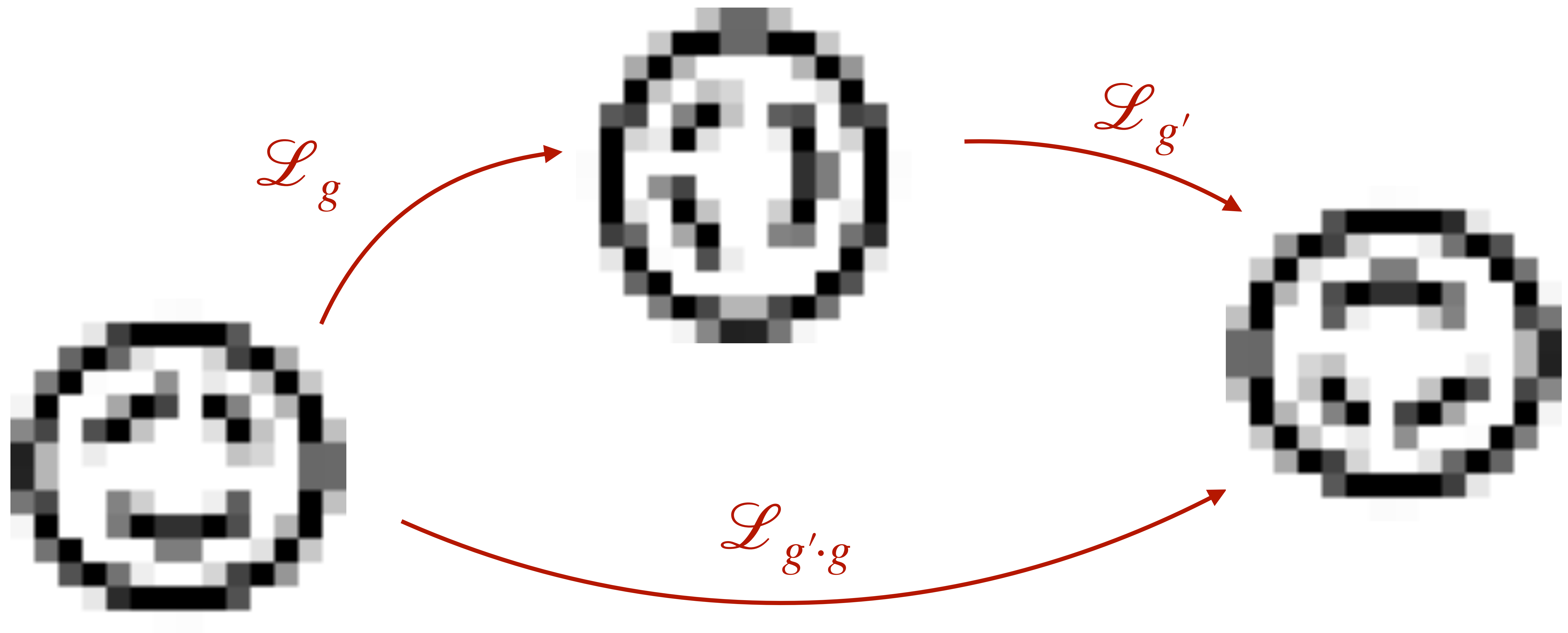
- a 2D image

$G = SE(2)$

- the roto-translation group

$\mathscr{L}_g(f)(\mathbf{y}) = f(\mathbf{R}_\theta^{-1}(\mathbf{y} - \mathbf{x}))$

- a roto-translation of the image



A left-regular representation $\mathscr{L}_g$ is a representation that transforms functions f by transforming their domains via the inverse group action

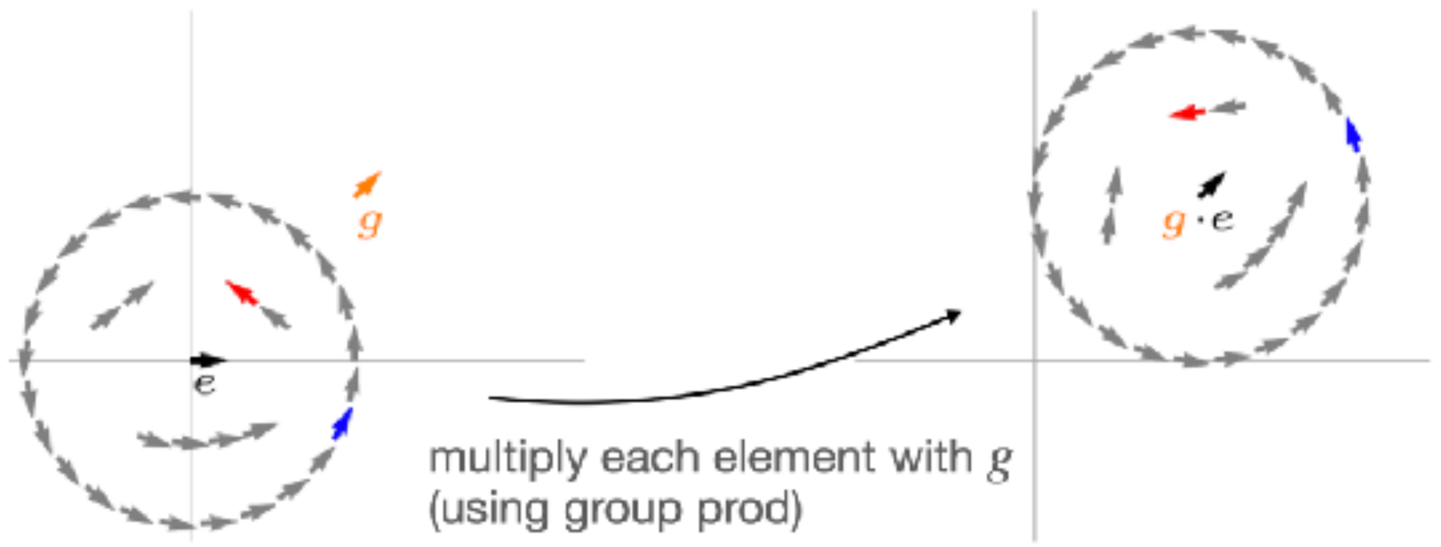
$$\mathscr{L}_g[f](x) := f(g^{-1} \cdot x)$$

"group action" equals group product when domain is G

Group actions

Group product (the action on G)

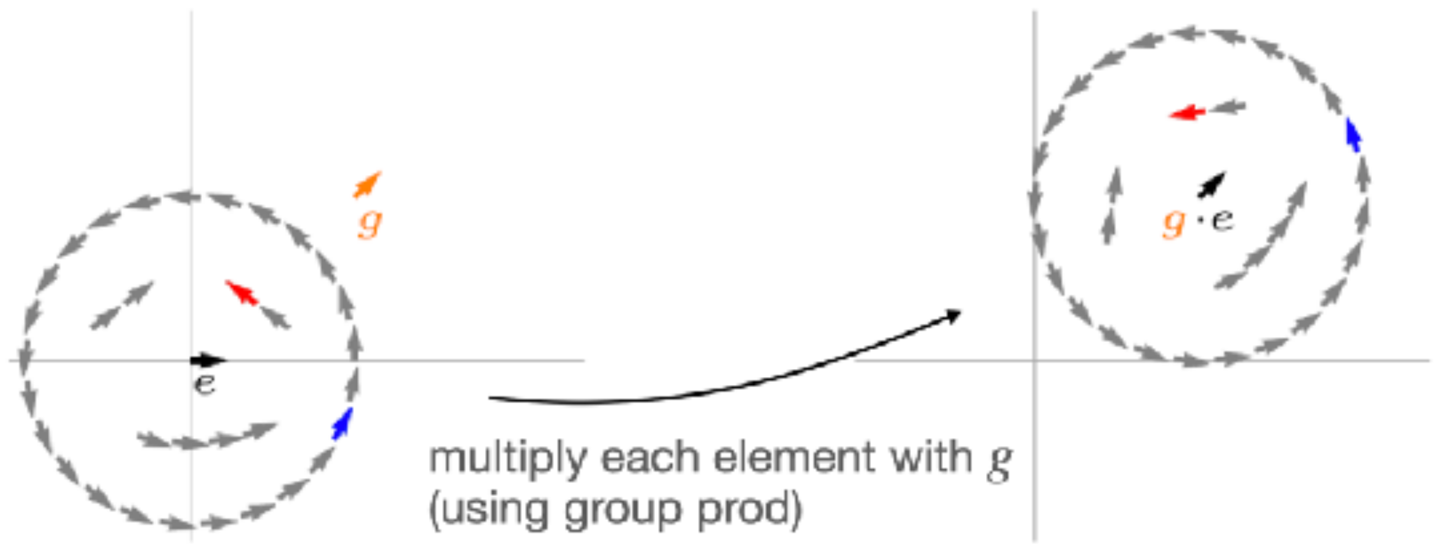
$$g \cdot g'$$



Group actions

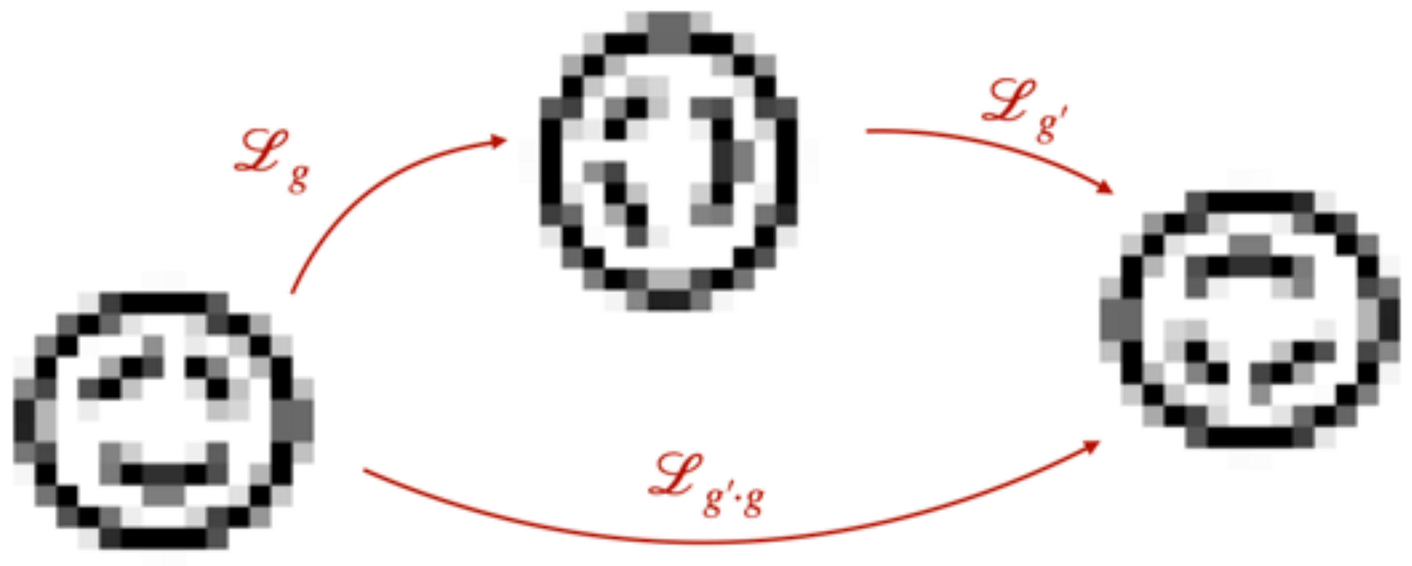
Group product (the action on G)

$$g \cdot g'$$



Left regular representation (the action on $\mathbb{L}_2(X)$)

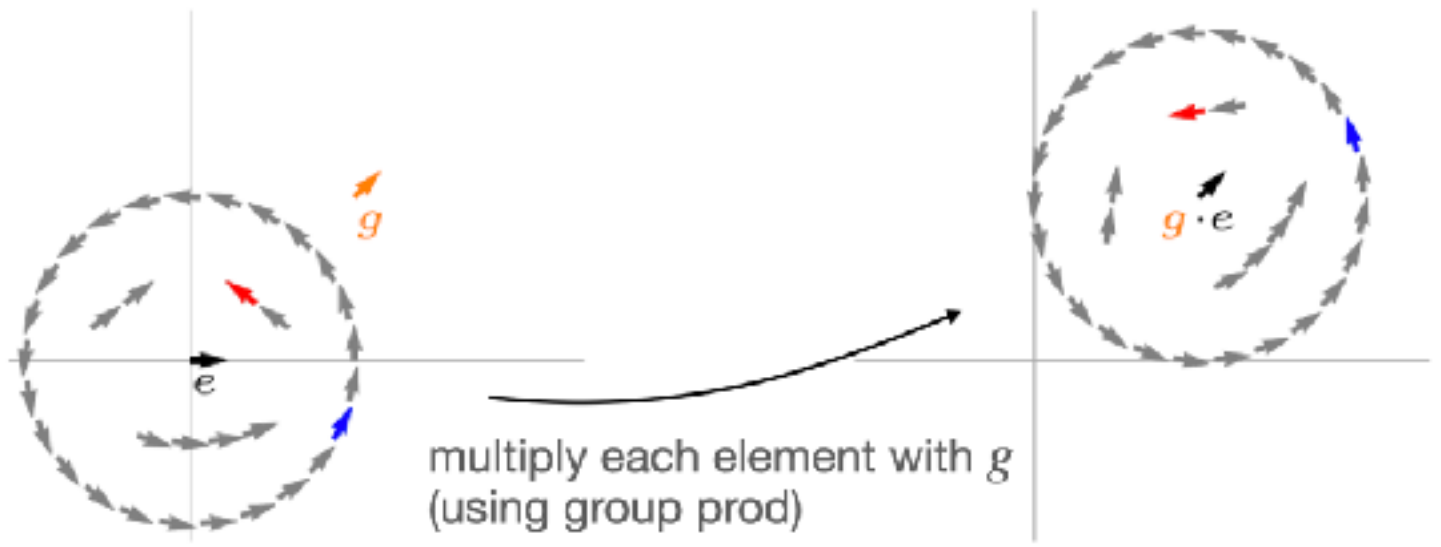
$$\mathscr{L}_g f$$



Group actions

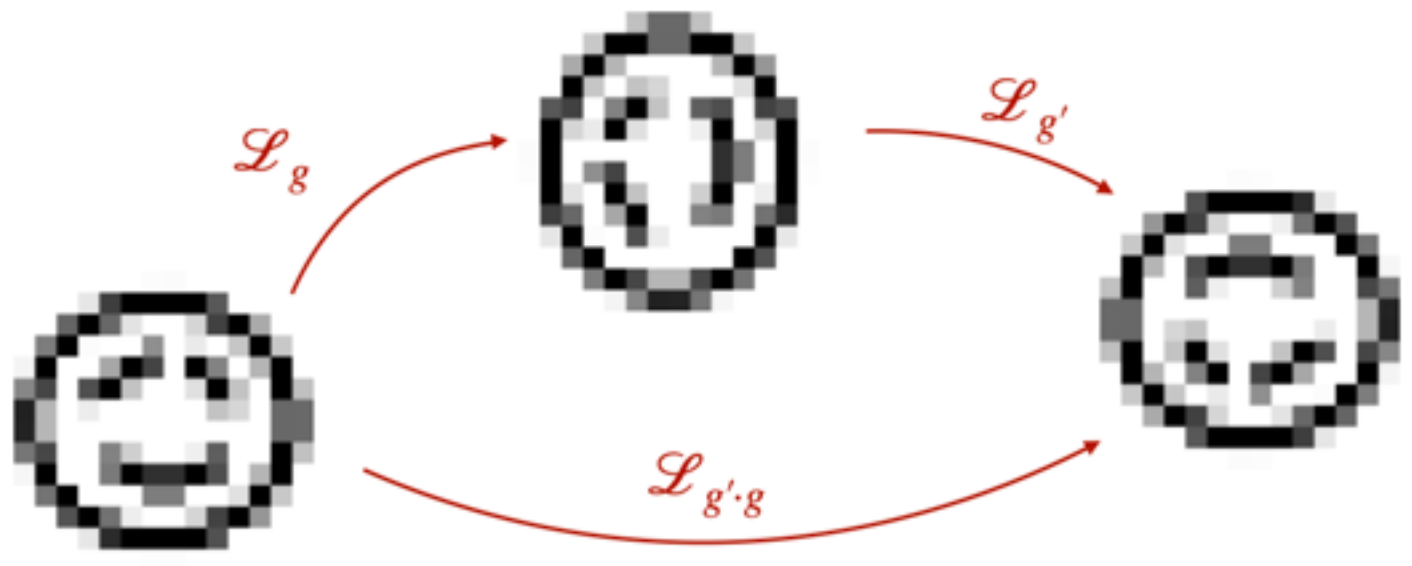
Group product (the action on G)

$$g \cdot g'$$



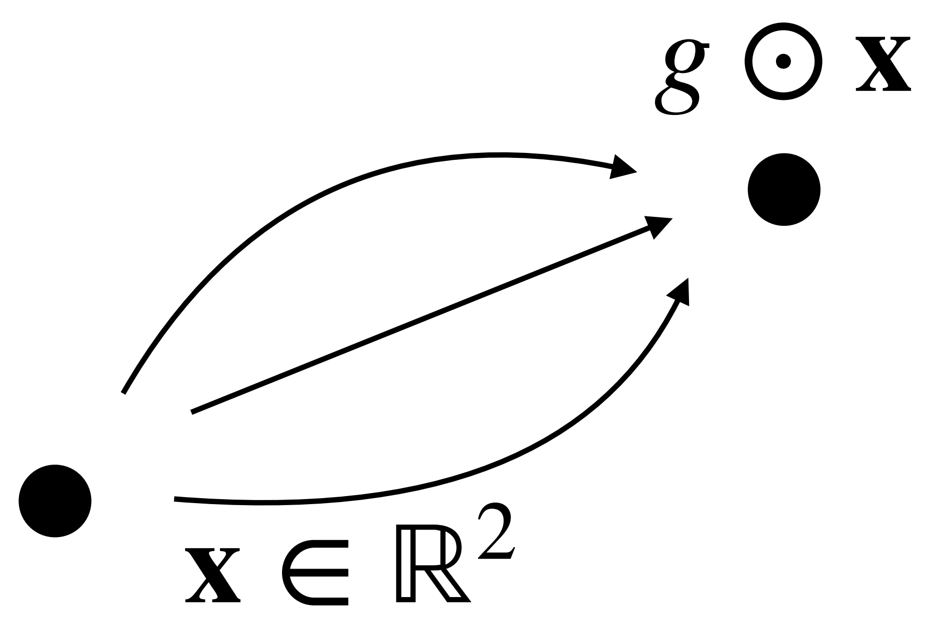
Left regular representation (the action on $\mathbb{L}_2(X)$)

$$\mathscr{L}_g f$$



Group action (the action on $\mathbb{R}^d$)

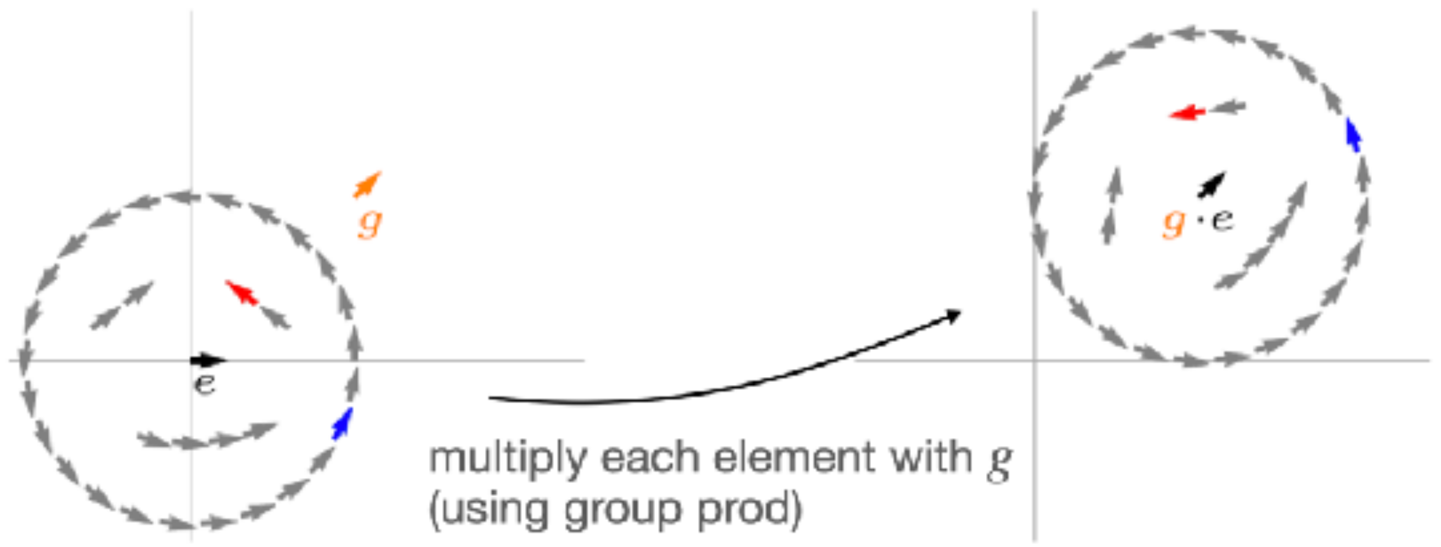
$$g \odot \mathbf{x}$$



Group actions

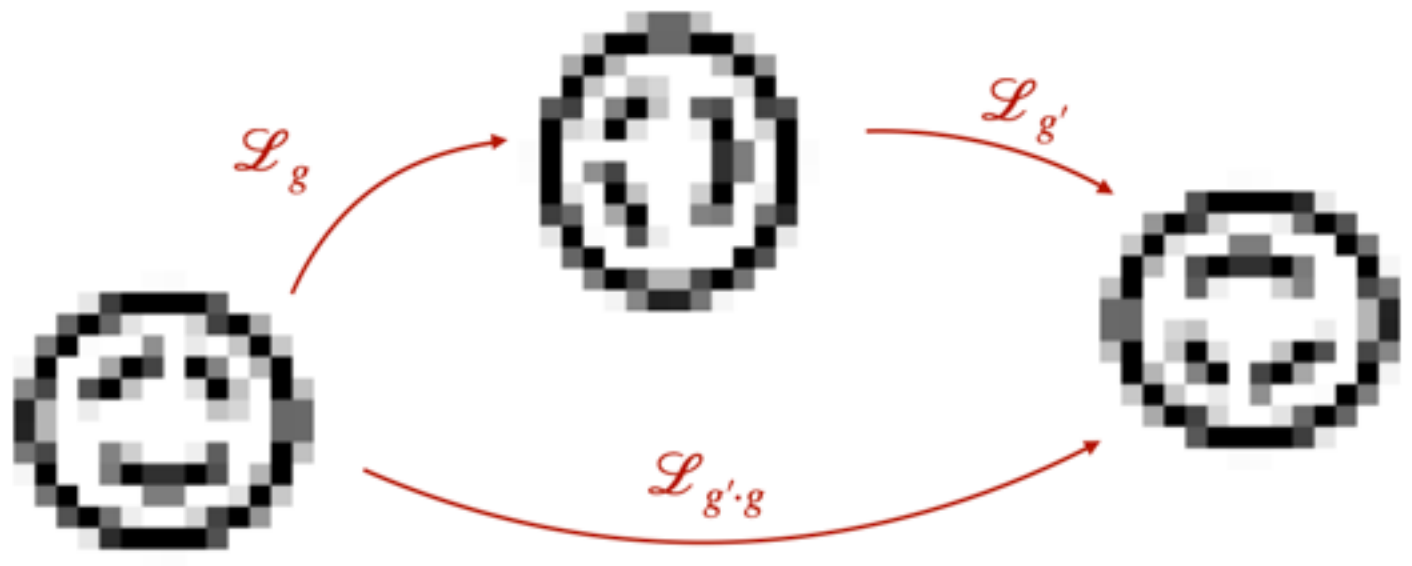
Group product (the action on G)

$g \cdot g'$ $\quad$ gg'



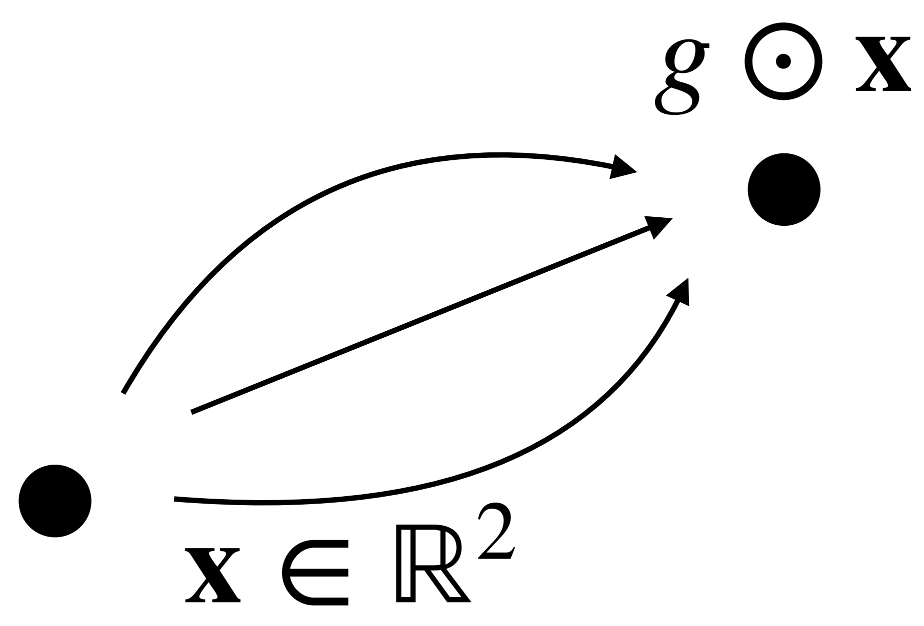
Left regular representation (the action on $\mathbb{L}_2(X)$)

$\mathscr{L}_g f$ $\quad$ gf



Group action (the action on $\mathbb{R}^d$)

$g \odot \mathbf{x}$ $\quad$ $g\mathbf{x}$



Equivariance

Equivariance is a property of an operator $\Phi : X \to Y$ (such as a neural network layer) by which it commutes with the group action:

$$\Phi \circ \rho^X(g) = \rho^Y(g) \circ \Phi$$

group representation action on X